Introduction to the mathematics of quantum economics

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(1) Use mathematics as a shorthand language, rather than an engine of inquiry. (2) Keep to them till you have done. (3) Translate into English. (4) Then illustrate by examples that are important in real life. (5) Burn the mathematics.
Alfred Marshall, 1906[2]

Too large a proportion of recent "mathematical" economics are merely concoctions, as imprecise as the initial assumptions they rest on, which allow the author to lose sight of the complexities and interdependencies of the real world in a maze of pretentious and unhelpful symbols.
John Maynard Keynes, 1936[3]

Nature isn't classical, dammit, and if you want to make a simulation of nature, you'd better make it quantum mechanical.
Richard Feynman, 1981[4]

1. Introduction

This document gives a technical introduction to some of the mathematics used in quantum economics, and is intended as a supplement for the book *Quantum Economics: The New Science of Money*. As the quotes above point out, economics is not the same as a mathematical proof, and the key ideas of quantum economics, such as the quantum theory of money and value, do not rely on equations. However the quantum formalism *is* mathematical, so to fully exploit its ideas some mathematics is useful (even if it is burned afterwards). The aim here is to sketch out the way in which the economy can be represented mathematically using the quantum formalism, show the advantages over the classical approach, and clarify

[1] Please cite as: Orrell, D. (4 February 2019), *Introduction to the mathematics of quantum economics*. Retrieved from http://www.postpythagorean.com/quantumeconomicsmath.pdf
[2] "(6) If you can't succeed in 4, burn 3. This last I do often." Letter to A.L. Bowley, 27 February 1906.
[3] From *The General Theory of Employment, Interest and Money*.
[4] From a 1981 talk "Simulating Physics with Computers" on the idea of a quantum computer.

(at least for those with some knowledge of basic matrix algebra) what it means to say that the economy can be treated as a quantum system in its own right.

The quantum approach to economics is inspired by the empirical fact that the monetary system shows quantum properties such as discreteness, indeterminacy, entanglement, and so on. To borrow Feynman's expression, a simulation had therefore better be quantum mechanical too, in the sense that it reflects these properties (even if it doesn't directly use a quantum formalism). The point is therefore not that the quantum approach will be the best technique to model every aspect of the economy, but rather that the economy has quantum properties which may need to be taken into account (explicitly or implicitly) depending on the context.

As an example from physics, weather forecasters do not base their models on quantum mechanics, but they do base them on the complex properties of water, which emerge from quantum mechanics. So in this case the main lesson is that the quantum properties of water molecules lead to highly complex emergent properties at the global level, which resist reduction to a lower level. In economics, one might therefore conclude that economic behaviour should be modelled at the appropriate level, so quantum properties are not directly relevant. On the other hand, in physics a technology such as an atom bomb scales quantum properties up to the macro level. Similarly, money is a designed technology, and its properties sometimes scale up to affect the economy as a whole, for example through phenomena such as money creation by private banks.

Models are ultimately justified by their success at explaining and predicting data. While the focus here is on presenting the basic tools of the theory, and showing how they relate to the nature of economic transactions, rather than on specific results, it should be noted that the areas of quantum cognition and quantum finance are heavily empirical, basing their results on experimental data for the former, and market data for the latter. The broader area of quantum economics – dealing as it does with emergent properties of a complex system – incorporates in addition a variety of complexity-based techniques, from agent-based models to systems dynamics, which have also been empirically tested (an exception is quantum agent-based models, which to my knowledge have yet to be developed for economics). The most obvious empirical argument for the quantum approach, though, is simply the nature of money, which is designed to have quantum properties.[5] For details, please see the book, and the references therein.

An outline is as follows. Section 2 introduces the idea of the Hilbert space, and shows how quantum probability differs from its classical version using the example of human cognition. Section 3 discusses the quantization procedure for a dynamic system. In Section 4 this is applied to the paridigmatic example of the quantum harmonic oscillator, and it is shown how tools such as creation and annihilation operators can be used to build up a system of interacting bosons. Section 5 uses the same ideas to develop a quantum model of a market, where shares and cash now take the place of bosons. Section 6 explores the quantum representation of supply and demand. Section 7 extends this dynamical analysis to production and consumption, and points towards how one could construct quantum models for more general applications. Section 8 discusses the concept of entanglement, and Section 9 summarises the main conclusions.

2. Some basics

Perhaps the most basic mathematical tool in quantum theory is the concept of the Hilbert space, which is named for the German mathematician David Hilbert (1862-1943). It was developed as an abstract mathematical object in the first decade of the twentieth century, and was later adopted by researchers in quantum physics. Social scientists are now following their lead by applying it to problems in areas such as decision-making and finance, as seen below.[6]

A Hilbert space H is a type of vector space whose elements, denoted $|u\rangle$, have coefficients that can be complex numbers. The dual state $\langle u|$ is the complex conjugate of the transpose of $|u\rangle$. The inner product between two elements $|u\rangle$ and $|v\rangle$ is denoted $\langle u|v\rangle$, and is analogous to the dot product in a normal vector space, with the difference that the result can again be complex. The outer product is denoted $|u\rangle\langle v|$, and is like multiplying a column vector by a row vector, which yields a matrix. The magnitude of an element $|u\rangle$ is given by $\sqrt{\langle u|u\rangle}$, and

[5] See: Orrell, D. (2018), 'Quantum Economics', *Economic Thought*, 7 (2).

[6] Some researchers in cognitive science prefer to treat the Hilbert space as just a tool, and see the word "quantum" as a distraction. Irving Fisher, in his 1892 book *Mathematical Investigations in the Theory of Value and Prices*, had a similar problem with the word "utility" which he described as "the heritage of Bentham and his theory of pleasures and pains. For us his *word* is the more acceptable, the less it is entangled with his *theory*" (p. 23). Personally I think it would be a little forced to ignore the theory's connections with physical reality.

two elements are orthogonal if $\langle u|v\rangle = \langle u|v\rangle = 0$. The Hilbert space can therefore be viewed as a generalisation of Euclidean space, with the difference that there can be an infinite number of dimensions (though conditions apply), the basis need not be simple column vectors, and coefficients can be complex.

An operator $\hat{A}$ is a map which sends one element $|u\rangle$ of H to another element $\hat{A}|u\rangle$ of H. For example, the projection operator is defined as $\hat{P}_u = |u\rangle\langle u|$, and $\hat{P}_u|v\rangle = |u\rangle\langle u|v\rangle$ gives the projection of v onto u. Operators $\hat{A}$ and $\hat{B}$ do not generally commute, so $\hat{A}\hat{B} \neq \hat{B}\hat{A}$. A state $|u\rangle$ is an eigenvector of $\hat{A}$ if $\hat{A}|u\rangle = \lambda|u\rangle$ where λ is the associated eigenvalue. For example $\hat{P}_u|u\rangle = |u\rangle\langle u|u\rangle = \lambda|u\rangle$, so $|u\rangle$ is an eigenvector of $\hat{P}_u$ with eigenvalue $\lambda = \langle u|u\rangle$. The expectation value of a linear operator $\hat{A}$ in the state $|u\rangle$ is given by $\langle u|\hat{A}|u\rangle$, i.e. the scalar product of $\langle u|$ with $\hat{A}|u\rangle$.

A key feature of quantum theory is that observables such as a particle's position or momentum are represented by Hermitian operators, which have real eigenvalues.[7] Instead of being passive elements, as in classical theory, they are operators that ask a question of the system. During a measurement of an observable, the system state $|S\rangle$ collapses to one of the eigenvectors of the associated operator, with a probability given by the square of the projection of the state $|S\rangle$ on that eigenvector.

To see the difference between the classical and quantum approaches, in the context of human cognition, suppose that a person has a choice between a certain number of possible options. In classical probability theory, each choice u would be treated as a subset of the set U consisting of all choices. A person's cognitive state is represented by a function p with the probability of choosing X given by $p(u)$. As a simple example, U could consist of two choices u and v, with respective probabilities $p(u)$ and $p(v)$, that satisfy $p(u) + p(v) = 1$.

In quantum cognition, a choice in response to a particular question is treated instead as an element (e.g. vector) $|u\rangle$ of a Hilbert space H, and a person's cognitive state is represented by an element $|S\rangle$, both of length 1. (The state $|S\rangle$ is sometimes called a wave function, although here it is static rather than time-varying.) Here the associated operator $\hat{P}_u$ is the one that

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[7] A Hermitian operator is one which equals its Hermitian conjugate, which for a matrix operator is defined as the complex conjugate of the transpose, so $A = A^\dagger \equiv (A^T)^*$.

projects vectors onto the vector $|u\rangle$. The probability of the answer to the question being $|u\rangle$ is then given by the magnitude of the projection squared, which is $|\langle u|S\rangle|^2$.

As a simple example, the two axes in the figure below represent decisions of Yes or No to some question, while a person's state is represented by the grey line at an angle α to the No axis. The probability of deciding Yes is given by the square of the projection onto the Yes axis, which equals $\sin^2 \alpha$.

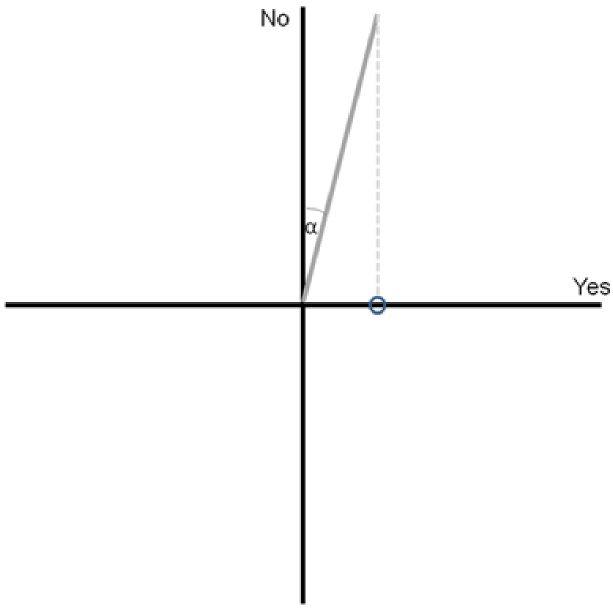

Figure 2.1. Axes show decision states which correspond to eigenvectors, grey line shows a person's state $|S\rangle$*. The probability of deciding Yes is found by projecting onto the Yes axis and taking the square, which gives* $\sin^2 \alpha$.

This shift, from sets of elements to geometric projections, allows for more complicated probabilistic effects such as non-commutativity and interference, which are characteristic of human cognition. For example, projecting onto $|u\rangle$, and then onto $|v\rangle$, may not give the same result as when the order is reversed, which compares with the "order effect" in surveys.[8] For a worked example, see the book's Appendix, or the web application available at https://david-systemsforecasting.shinyapps.io/ordereffect/ (see screenshot below). The Hilbert space

[8] For the 2-D case the coefficients can be assumed to be real rather than complex, see Moreira, C. & Wichert, A. (2017), 'Are Quantum Models for Order Effects Quantum?', *International Journal of Theoretical Physics* 56(12): 4029–4046.

therefore appears to be the natural framework for simulating cognitive phenomena, and researchers have amassed a considerable number of empirical findings to back up that claim.[9]

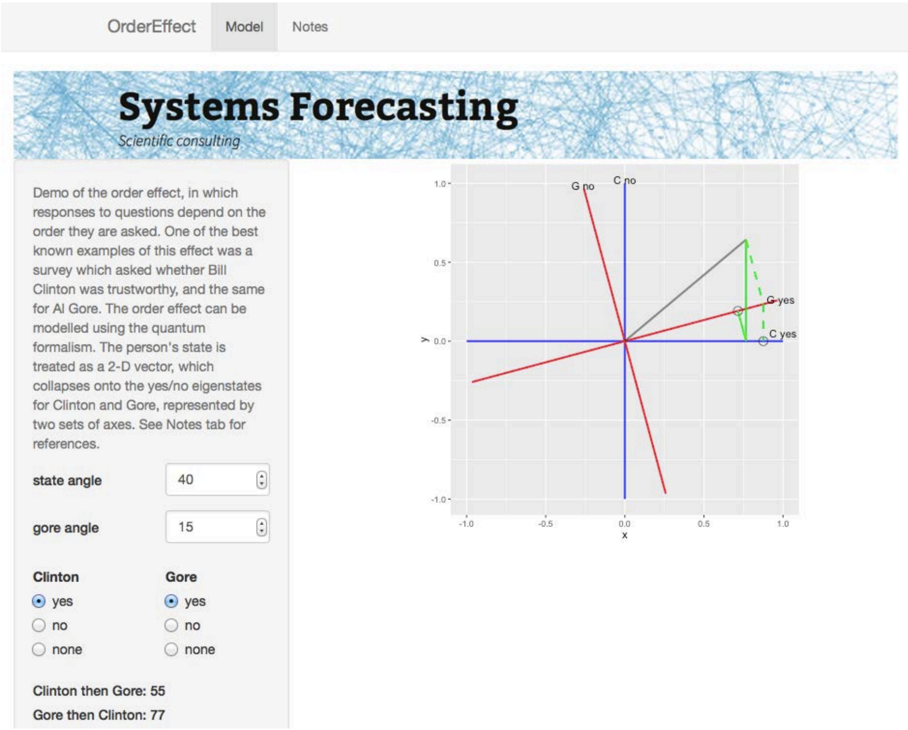

Figure 2.2. Screenshot of the order effect demo, available as a web application.

3. Quantizing dynamics

As seen above, the key idea in the quantum approach is that point objects are replaced with quantum state or wave functions, and observables are replaced with the eigenvalues of operators. A typical question in physics is how a given set of equations can be “quantized” in this way.

One clue on how to go about this is the fact (first discovered by the mathematician Oliver Heaviside in the late nineteenth century) that differential operators act in some respects like ordinary numbers. Consider for example the equation

$$y + \frac{dy}{dx} = x^2.$$

[9] For a survey, see: Bruza, Peter D. et al. (2015), ‘Quantum cognition: a new theoretical approach to psychology’, *Trends in Cognitive Sciences* 19(7), pp 383 – 393.

Define D to be the differential operator $D = \frac{d}{dx}$, so $Dy = \frac{dy}{dx}$. Powers of D are interpreted as higher derivatives, so

$$D^2 = \frac{d^2}{dx^2}$$

and so on. Then the above equation can be written

$$(1 + D)y = x^2$$

so

$$y = \frac{x^2}{1 + D}.$$

Rewriting $\frac{1}{1+D}$ as the infinite expansion

$$\frac{1}{1 + D} = 1 - D + D^2 - D^3 \dots$$

gives

$$y = (1 - D + D^2 - D^3 \dots)x^2 = x^2 - 2x + 2$$

after applying the derivative operators to x and noting that all derivatives higher than the second are zero.

Because operators act on the object to the right of them, the two don't usually commute. Suppose we have a function $\psi(x)$ and evaluate

$$D(x\psi) = D(x)\psi + xD(\psi) = \psi + xD(\psi)$$

so

$$D(x\psi) - xD(\psi) = D(x)\psi + xD(\psi) = \psi$$

or in operator form

$$Dx - xD = 1$$

where 1 is the identity operator that does nothing. The commutator for two elements f and g is defined as $[f, g] = fg - gf$, so here we can write $[D, x] = 1$. Such commutator relationships play an important role in quantum mechanics. One has to be careful about the order of operations, and in quantizing a system it may not be clear at first which is the correct order to use.

Now, we want to represent quantum states using wave functions. Many experiments suggest waves that have a periodicity which scales with momentum, in a manner which depends on the reduced Planck's constant $\hbar$. Focussing on the spatial variation, a typical wave function might therefore be of the form

$$\psi(x) = e^{-\frac{ipx}{\hbar}}.$$

In classical mechanics x would refer to a spatial coordinate, and p to a momentum. If we identify $\hat{p}$ as the differential operator

$$\hat{p} = -i\hbar\frac{\partial}{\partial x}$$

and apply it to ψ we get

$$\hat{p}\psi = -i\hbar\frac{\partial\psi}{\partial x} = \hat{p}e^{-\frac{ipx}{\hbar}} = p\psi$$

so the observable p is an eigenvalue of the operator. We can therefore identify $\hat{p}$ as the momentum operator. The position operator $\hat{x}$ returns the value of x. In "momentum space" it can be defined as

$$\hat{x} = i\hbar\frac{\partial}{\partial p}$$

which has the eigenvalue x. A similar relationship (related to the requirements of relativity) holds for total energy $\hat{H}$ and time t:

$$\hat{H} = -i\hbar\frac{\partial}{\partial t}.$$

Using the definition of the momentum operator, and the product rule of calculus, we have

$$\hat{x}\hat{p}\psi - \hat{p}\hat{x}\psi = \hat{x}\left(-i\hbar\frac{\partial\psi}{\partial x}\right) + i\hbar\frac{\partial(\hat{x}\psi)}{\partial x}$$

$$= -\hat{x}\left(i\hbar\frac{\partial\psi}{\partial x}\right) + i\hbar\left(\hat{x}\frac{\partial\psi}{\partial x} + \frac{\partial\hat{x}}{\partial x}\psi\right) = i\hbar\frac{\partial\hat{x}}{\partial x}\psi.$$

But since $\frac{\partial\hat{x}}{\partial x} = 1$, it follows that $\hat{x}\hat{p}\psi - \hat{p}\hat{x}\psi = i\hbar\psi$, and the commutator therefore satisfies the relationship $[\hat{x},\hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar$. This is known as the canonical commutator relationship, which holds also for other pairs such as energy and time.

To get a better sense of how the quantization procedure works, we can apply the method to a simple physical example, which is the harmonic oscillator.[10] We choose it because it plays a key role in quantum field theory, which underpins the methods used later to describe the quantum economy; it serves as a first-order approximation to many more complicated

[10] If the quantization procedure seems a little ad hoc and awkward, one reason is that we are trying to adapt classical mathematical tools to handle wave/particle duality. Another is that the approach was based on intuition and the equations were adopted, not because they can be proved to be true, but because they fit the data (which gives some latitude for social scientists to adapt them for other uses).

systems; and it is one of the few quantum systems that can be solved in closed form equations.

4. The harmonic oscillator

A classical harmonic oscillator involves an object of mass *m* oscillating around a central point with a spring-like restoring force given by $F = -kx$, where *k* is a constant and *x* is the displacement. The equation of motion can be written in terms of momentum *p* as

$$p = m\dot{x}$$
$$\dot{p} = F = -kx$$

or equivalently as $m\ddot{x} = -kx$. This has the oscillatory solution

$$x = A\cos(\omega t + \varphi)$$

where the phase φ depends on the starting point. The energy is given by

$$E = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

where $\omega = \sqrt{k/m}$ is the frequency of oscillation. The first term represents the kinetic energy, and the second term the potential energy.

To quantize the system, we again need to replace the classical equations with quantum versions that act on wave functions but recover the required properties of observables.[11] In quantum mechanics, the total energy is given by an equation known as the Hamiltonian, expressed now in terms of operators. We therefore try:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2.$$

This can be written more simply in the form

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) = \hbar\omega\left(\hat{N} + \frac{1}{2}\right)$$

where

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right),$$

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right),$$

$$\hat{N} = \hat{a}^\dagger\hat{a}.$$

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[11] I am drawing on: Barton Zwiebach. 8.05 Quantum Physics II. Fall 2013. Massachusetts Institute of Technology: MIT OpenCourseWare, https://ocw.mit.edu. License: Creative Commons BY-NC-SA.

For reasons that will become clear, $\hat{a}^\dagger$ is known as the creation operator, $\hat{a}$ is the annihilation operator, and $\hat{N}$ is the number operator. As seen by multiplying them out and using the commutator relationship between $\hat{x}$ and $\hat{p}$, the creation and the annihilation operators satisfy the canonical commutator relationship with this scaling, which is

$$[\hat{a}, \hat{a}^\dagger] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1.$$

If ψ is a wave function with norm 1, then

$$\langle\psi|\hat{H}|\psi\rangle = \hbar\omega\left\langle\psi\middle|\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)\middle|\psi\right\rangle = \hbar\omega\langle\hat{a}\psi|\hat{a}\psi\rangle + \frac{\hbar\omega}{2} \geq \frac{\hbar\omega}{2}$$

since any norm cannot be less than zero.

Now, suppose that $|E\rangle$ is a normalised energy state of the system. Since observables correspond to eigenvalues, it follows that $|E\rangle$ must be an eigenvector of the Hamiltonian operator, with associated eigenvalue E:

$$\hat{H}|E\rangle = E|E\rangle.$$

From this and the above inequality, we have

$$\langle E|\hat{H}|E\rangle = E\langle E|E\rangle = E \geq \frac{\hbar\omega}{2}.$$

The system therefore has a minimum energy level given by $\frac{\hbar\omega}{2}$.

Consider the two states defined as

$$|E_+\rangle = \hat{a}^\dagger|E\rangle,$$
$$|E_-\rangle = \hat{a}|E\rangle.$$

We first note that

$$[\hat{H}, \hat{a}^\dagger] = \hat{H}\hat{a}^\dagger - \hat{a}^\dagger\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a})\hat{a}^\dagger - \hat{a}^\dagger\hbar\omega(\hat{a}^\dagger\hat{a}) = \hbar\omega(\hat{a}^\dagger\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}^\dagger\hat{a})$$

since the contribution of the constant term in the Hamiltonian cancels out. Using the commutator relationship for creation and annihilation operators then gives

$$[\hat{H}, \hat{a}^\dagger] = \hbar\omega\hat{a}^\dagger[\hat{a}, \hat{a}^\dagger] = \hbar\omega\hat{a}^\dagger.$$

Similarly

$$[\hat{H}, \hat{a}] = \hbar\omega\hat{a}$$

and also

$$\hat{N}|E\rangle = \left(\frac{\hat{H}}{\hbar\omega} - \frac{1}{2}\right)|E\rangle = \hat{N}_E|E\rangle$$

where $\hat{N}_E = \frac{\hat{H}}{\hbar\omega} - \frac{1}{2}$ is the number operator eigenvalue associated with this energy state.

Then

$$\hat{H}|E_+\rangle = \hat{H}\hat{a}^\dagger|E\rangle = \left([\hat{H},\hat{a}^\dagger] + \hat{a}^\dagger H\right)|E\rangle = (\hbar\omega + E)\hat{a}^\dagger|E\rangle = (E + \hbar\omega)|E_+\rangle,$$

$$\hat{H}|E_-\rangle = \hat{H}\hat{a}|E\rangle = \left([\hat{H},\hat{a}] + \hat{a}H\right)|E\rangle = (-\hbar\omega + E)\hat{a}|E\rangle = (E - \hbar\omega)|E_-\rangle$$

so the energy state with $E_+ = E + \hbar\omega$ and $N_{E_+} = N_E + 1$ has an increased energy level, while the energy state with $E_- = E - \hbar\omega$ and $N_{E_-} = N_E - 1$ has a decreased energy level.

The reason $\hat{a}^\dagger$ is called the creation operator, and $\hat{a}$ the annihilation operator, is that these operators raise or lower the energy by $\hbar\omega$ and the number operator by one. The creation operator can always be applied to raise the energy, but the annihilation operator can only be applied to energy levels above the base level, since energy cannot be negative.

The lowest base level can be found by assuming there is a non-trivial state $|E\rangle$ that is annihilated by $\hat{a}$, so $\hat{a}|E\rangle = 0$. Thus $\hat{a}^\dagger\hat{a}|E\rangle = N|E\rangle = 0$, which implies that this is the energy state with $E = \frac{\hbar\omega}{2}$ and $N_E = 0$. We can derive the equation for this state by acting with position *x*:

$$\langle x|\hat{a}|E\rangle = \sqrt{\frac{m\omega}{2h}}\left\langle x\middle|\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right)\middle|E\right\rangle = 0.$$

If we define the wave function $\psi_E(x) = \langle x|E\rangle$ and use the definition of $\hat{p}$ as a differential operator, then this gives

$$\left(x + \frac{\hbar}{m\omega}\frac{d}{dx}\right)\psi_E(x) = 0$$

or

$$\frac{d\psi_E}{dx} = -\frac{m\omega}{\hbar}x\psi_E$$

with solution

$$\psi_E(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x^2\right)$$

which is a Gaussian distribution centered at 0. The existence of this ground state reflects the uncertainty principle, in the sense that an oscillator with no energy can't exist (because then we would know the energy is zero), and has no classical analogue. Higher energy states are more complicated, and can be determined by successively applying the creation operator.

Another way to express this is by using the number operator. Denote states as $|n\rangle$ with associated eigenvalue n, so $\widehat{N}|n\rangle = n|n\rangle$. The ground state is $|0\rangle$ (which is not the same as the zero vector). The next state $|1\rangle$ is obtained by using the creation operator on $|0\rangle$,

$$|1\rangle = \hat{a}^{\dagger}|0\rangle$$

and

$$\widehat{N}|1\rangle = \hat{a}^{\dagger}\hat{a}\hat{a}^{\dagger}|0\rangle = (\hat{a}^{\dagger}[\hat{a},\hat{a}^{\dagger}] + \hat{a}^{\dagger}\hat{a}^{\dagger}\hat{a})|0\rangle = \hat{a}^{\dagger}|0\rangle = 1$$

where we have used $[\hat{a},\hat{a}^{\dagger}] = 1$ and $\hat{a}|0\rangle = 0$. The equations for higher energy states can be derived recursively to give

$$|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^{\dagger})^{n}|0\rangle.$$

The states $|n\rangle$ form an orthonormal basis, so any state can be described in terms of a linear combination of these states.

One can also calculate the expected values of quantities for different energy levels. Some algebra using the creation and annihilation operators shows that

$$\langle n|\hat{x}|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle n|\hat{a} + \hat{a}^{\dagger}|n\rangle = 0$$

$$\langle n|\hat{p}|n\rangle = i\sqrt{\frac{m\omega\hbar}{2}}\langle n|\hat{a}^{\dagger} - \hat{a}|n\rangle = 0$$

$$\langle n|\hat{x}^2|n\rangle = \frac{\hbar}{2m\omega}\langle n|(\hat{a} + \hat{a}^{\dagger})^2|n\rangle = \frac{\hbar}{m\omega}\left(n + \frac{1}{2}\right)$$

$$\langle n|\hat{p}^2|n\rangle = -\frac{m\omega\hbar}{2}\langle n|(\hat{a}^{\dagger} - \hat{a})^2|n\rangle = m\omega\hbar\left(n + \frac{1}{2}\right).$$

The uncertainties in position and momentum therefore satisfy

$$\Delta x\Delta p = \sqrt{\langle n|\hat{x}^2|n\rangle}\sqrt{\langle n|\hat{p}^2|n\rangle} = \frac{\hbar}{m\omega}\left(n + \frac{1}{2}\right) \geq \frac{\hbar}{2m\omega}$$

which is Heisenberg's uncertainty principle.

Another operator which will prove useful is the translation operator defined as

$$T_{x_0} = e^{-\frac{i}{\hbar}\hat{p}x_0}$$

which acts on a state $|\psi\rangle$ by moving it by an amount x_0. To see this, the expectation value of $\hat{x}$ in the state $|\psi\rangle$ is

$$\langle\hat{x}\rangle_{\psi} = \langle\psi|\hat{x}|\psi\rangle$$

and the expectation of $\hat{x}$ in the state $T_{x_0}|\psi\rangle$ is

$$\langle\hat{x}\rangle_{T_{x_0}|\psi} = \langle\psi|T_{x_0}^{\dagger}\hat{x}T_{x_0}|\psi\rangle = \left\langle\psi\left|e^{-\frac{i}{\hbar}\hat{p}x_0}\hat{x}e^{\frac{i}{\hbar}\hat{p}x_0}\right|\psi\right\rangle.$$

The expression involving brackets can be solved to give

$$\langle\hat{x}\rangle_{T_{x_0}|\psi} = \left\langle\psi\left|\hat{x} + \frac{i}{\hbar}[\hat{p},\hat{x}]x_0\right|\psi\right\rangle = \hat{x} + x_0$$

as expected.[12]

If the translation operator is applied to the ground state $|0\rangle$, then the new state is called a coherent state, and can be expressed in terms of creation and annihilation operators as follows:

$$|\hat{x}_0\rangle = T_{x_0}|0\rangle = \exp\left(-\frac{i}{\hbar}\hat{p}x_0\right)|0\rangle = \exp\left(\frac{x_0}{\sqrt{2}d}(\hat{a}^{\dagger} - \hat{a})\right)|0\rangle,$$

or alternatively

$$|\alpha\rangle = \mathrm{D}(\alpha)|0\rangle$$

where $d = \sqrt{\frac{\hbar}{m\omega}}$ is a length scale, $\alpha = \frac{x_0}{\sqrt{2}d}$, and

$$\mathrm{D}(\alpha) = \exp(\alpha\hat{a}^{\dagger} - \alpha^*\hat{a})\,|0\rangle$$

is known as the displacement operator. When α is real, as here, the displacement is in position only, while imaginary values correspond to displacement in momentum.

Calculation shows that the total energy of the translated system is increased relative to that of the ground state by an amount $\frac{1}{2}m\omega^2{x_0}^2$ which makes sense since it corresponds to the classical potential energy of a particle on a spring stretched an amount x_0. However the system is not in a single energy state, but is of the form

$$|\hat{x}_0\rangle = \sum_{n=0}^{\infty} c_n|n\rangle.$$

The probability of obtaining an energy equal to E_n is $c_n^2 = \frac{\lambda^n}{n!}e^{-\lambda}$ which is a Poisson distribution with mean $\lambda = \frac{m\omega x_0^2}{2\hbar}$.

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[12] Using the Baker-Hausdorff identity $e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A},\hat{B}] + \frac{1}{2!}[\hat{A},[\hat{A},\hat{B}]] +$ ... where here all but the first two terms vanish.

So far we have only viewed the system in a static sense. To study how the wave function $|\psi\rangle$ evolves with time, we write

$$|\psi\rangle_t = \hat{U}(t, t_0)|\psi\rangle_{t_0}$$

where $\hat{U}(t, t_0)$ is a unitary linear operator, that can be viewed as rotating the hyperspace of all possible states in the Hibert space. Taking the derivative with respect to time gives

$$\frac{\partial}{\partial t}|\psi\rangle_t = \frac{\partial \hat{U}(t, t_0)}{\partial t}|\psi\rangle_{t_0}.$$

Using the fact (easily checked) that

$$\hat{U}(t_0, t) = \hat{U}^{-1}(t, t_0) = \hat{U}^{\dagger}(t, t_0)$$

then gives

$$\frac{\partial}{\partial t}|\psi\rangle_t = \frac{\partial \hat{U}(t, t_0)}{\partial t}\hat{U}^{\dagger}(t, t_0)|\psi\rangle_t.$$

Recalling that

$$\hat{H} = -i\hbar\frac{\partial}{\partial t}$$

gives the Schrödinger equation

$$i\frac{\partial}{\partial t}|\psi\rangle_t = \hat{H}(t)|\psi\rangle_t$$

which can be solved in a similar manner as the classical version to show that operators satisfy the same oscillatory equations of motion.

To summarise, the quantum model predicts that the observed energy levels of a harmonic oscillator are equally spaced with an interval of $h\omega$ and a minimum value of $\frac{h\omega}{2}$. Prior to measurement, the system will be in a superposed state of the form $|\psi\rangle = \sum_n A_n|n\rangle$, where the A_n are complex numbers, and $w_n = |A_n|^2$ is the probability that the oscillator is in the state $|n\rangle$. The evolution of the state can be solved using the Schrödinger equation.

As a physical example of the harmonic oscillator, a diatomic molecule such as the hydrogen molecule H_2 can be viewed as two atoms connected by a spring. Experimental observations show that such molecules absorb and emit photons whose frequencies are multiples of the oscillator frequency, as expected. Many other physical systems, such as the vibration of molecules in a solid, can be similarly approximated as a system of quantum harmonic oscillators, since the assumption of a linear force can be viewed as a first-order estimate to the dynamics near the minima of a potential well. Most importantly, it turns out that the equations describing electromagnetic fields in quantum physics are like those of the harmonic oscillator, with the particles corresponding to photons, and the ground state corresponding to the energy of empty space. It is this energy that fuels the appearance of “virtual photons” which communicate the electromagnetic force. In economics, as seen below, a version of the oscillator can also be used to simulate the dynamics of supply and demand, and is a staple of quantum finance.

What carries over in a more general sense is the idea of representing a quantum system as a collection of particles, that can be added, removed, or translated through the use of operators. Indeed, another interpretation of the quantum model – known as the Fock space representation – is to see the harmonic oscillator as representing, not a single particle, but a collection of *n* fictitious particles each with energy $\hbar\omega$. In this picture, the creation and annihilation operators are seen as adding and removing these particles. The ground or vacuum state $|0\rangle$ has no particles, $|1\rangle$ has a single particle, $|2\rangle$ has two, and so on. This method, known as second quantization, underpins the quantum field theory of relativistic particles, used for example to represent systems of bosons. As seen in the next section it can also be applied to things like assets, where here *n* refers to the number of units held.

The other thing which carries over to economics is the different nature of classical and quantum systems. While the classical harmonic oscillator is just a weight bouncing around on a spring, where quantities such as position, momentum, and energy can be precisely calculated, the quantum version is better described in terms of potentiality. We can only calculate the probability that a measurement will yield a particular result; and the complexity of quantum behaviour means that even this can only be easily done for relatively simple systems. In economics, this puts a strong limit on how much can be gained from using reductionist methods.

5. The quantum market

In the examples above we have seen that a person’s cognitive state, or the state of a quantum harmonic oscillator, can be simulated as a member of a Hilbert space. Furthermore, single particles that are in superposed states can be viewed, in a dual sense, as a collection of fictitious particles in single states. We can do something similar for the economy as a whole, and model it as a collection of interacting particles in a Hilbert space. As a starting point, we will consider a simplified financial market. I will follow here the approach described by the late Rutgers theoretical physicist Martin Schaden in a 2002 paper on quantum finance, see that paper for details and applications.[13]

Suppose that the market consists of a collection of agents (investors) j = 1, 2, ..., J who buy and sell assets of types i = 1, 2, ..., I. Each agent holds cash (or debt) x^j. The market can be represented as a Hilbert space H, with the basis

$$B := \{|x^j, \{n_i^j(s) \geq 0, i = 1, .., I\}, j = 1, .., J\rangle\}.$$

Here $n_i^j(s)$ is the number of assets i with a price of s dollars that are held by investor j.

An individual basis state represents a market where the price of every security, and the cash position of each agent, is known precisely. The basis states are orthogonal in the sense that if the market is in the state $|m\rangle$ then it cannot be in a different state $|n\rangle$, so if $m \neq n$ then the inner product $\langle m|n\rangle = 0$. In general the market state (wave function) M is never known this accurately and is instead represented by the linear superposition of basis states $|n\rangle$ in B:

$$|M\rangle = \sum_n A_n |n\rangle$$

where the A_n are complex numbers, and $w_n = |A_n|^2$ is the probability that the market is in the state $|n\rangle$.

The phases of the A_n are left unspecified at this stage, but are key to understanding effects such as interference. As in quantum physics, these effects are seen more easily when considering individual transactions. The propensities of each agent to buy or sell an asset can themselves be modelled as quantum phenomena, which as already discussed experience interference effects, and these can interact to affect the market as a whole. We return to this below.

If we define the ground state $|0\rangle$ to be a market where agents hold no assets including cash, then we can build up a real market by transferring cash and assets to agents. The approach is the same as that used in many-body quantum mechanics to simulate the behaviour of a collection of bosons, so shares are added or removed from an agent's account by the use of

[13] Schaden, M. (2002), 'Quantum finance', *Physica A* 316(1), pp. 511-538.

the creation operator $\hat{a}_i^{\dagger j}(s)$ and the annihilation operator $\hat{a}_i^j(s)$. Money creation is handled using a translation operator of the form

$$\hat{c}^{\dagger j}(s) = \exp\left(-s\frac{\partial}{\partial x^j}\right)$$

which increases the amount of cash held by agent *j* by *s* currency units. Similarly the Hermitian conjugate operator $\hat{c}^j(s) = \hat{c}^{\dagger j}(-s)$ lowers the cash holding of agent *j* by the amount *s*.

While it might not be obvious from these dry equations, and we haven't considered factors such as the creation of money objects through the issuance of debt, money still has a very special (but usually understated) role in the quantum model. Unlike other assets, it has a stable defined price. Without money, it is impossible to assign a price to other assets in the first place. The fact that these assets have indeterminate value is what gives money its dualistic properties, combining as it does stable numbers and unstable values. While it isn't possible for an asset to have a negative price, an agent can have a negative amount of money. Finally, money is often created in the first place through loans, which lead to entanglement as discussed below.

The buying and selling of one unit of an asset by agent *j* at price *s* is represented by the creation and annihilation operators in combination with cash transfers which reflect the exchange of money:

$$\hat{b}_i^{\dagger j}(s) = \hat{a}_i^{\dagger j}(s)\hat{c}^j(s),$$

$$\hat{b}_i^j(s) = \hat{a}_i^{\dagger j}(s)\hat{c}^{\dagger j}(s).$$

We can build up an arbitrary market state from the vacuum state by using these operators to successively transfer cash and securities to each agent. To study how the market wave function evolves with time, we write

$$|M\rangle_t = \hat{U}(t, t_0)|M\rangle_{t_0}$$

where $\hat{U}(t, t_0)$ is a unitary linear operator. The dynamical behaviour of the system is driven by a Hamiltonian $\hat{H}(t)$, which again satisfies the Schrödinger equation

$$i\frac{\partial}{\partial t}|M\rangle_t = \hat{H}(t)|M\rangle_t.$$

It is then possible to develop Hamiltonians for things like cash flow, the trading of securities, and so on (although the mathematics is usually more complicated than for something like the harmonic oscillator). As shown by Schaden and other researchers, these in turn can be used to derive statistical properties of markets.

The variables of the system can be loosely interpreted in terms of physical analogies. The price *s* of an asset (or more correctly its logarithm) is like position. As in physics, there is an uncertainty relation involving asset price, and the momentum of the price change. The creation of money or assets adds energy (as measured by the Hamiltonian) to the total energy of the system. The same techniques used to study many-body quantum systems can then be applied to make predictions about market behaviour, either in closed form or by explicitly modelling each agent.

As a simple example of a Hamiltonian in finance, consider the case of a savings instrument containing an initial amount of cash x_0 which accumulates at an interest rate r. The classical Hamiltonian for this system is

$$H = rxq$$

where (in classical notation) q is the conjugate variable of x.[14] We then have

$$\frac{dx}{dt} = \frac{\partial H}{\partial q} = rx$$

$$\frac{dq}{dt} = -\frac{\partial H}{\partial x} = -rq.$$

Solving then gives

$$x = x_0 e^{rt}$$

$$q = q_0 e^{-rt}$$

which implies that the Hamiltonian is constant in time:

$$H = rxq = rx_0 e^{rt} q_0 e^{-rt} = rx_0 q_0.$$

Note that changing q_0 doesn't affect the result for *x*, so we can set $q_0 = 1$ which means that $q = e^{-rt}$ is the value of one unit of currency discounted to time $t = 0$. The units of the Hamiltonian are then units of currency (e.g. dollars).

To quantize the system, we again replace the Hamiltonian *H* and classical variables *x* and *p* with operators. Because the Hamiltonian must be Hermitian, we need to write it in a symmetric form as

[14] See e.g. Bensoussan, A., Chutani, A. & Sethi, S. (2009), 'Optimal Cash Management under Uncertainty', *Operations Research Letters* 37:425-429.

$$\hat{H} = \frac{r}{2}(\hat{x}\hat{q} + \hat{q}\hat{x}).$$

Standard techniques can then be used to show that the probability distribution of the cash holdings matches that expected from the classical case (as Schaden notes, the quantum approach only comes into its own when future returns are uncertain). One can draw an analogy with the Hamiltonian of a multi-boson system $\hat{H} = \hbar\omega\left(\hat{N} + \frac{1}{2}\right)$. The interest rate *r*, which like ω has units of inverse time, plays the role of frequency (another way to see it is as frequency of a fixed payment), while the initial investment plays the role of the number operator $\hat{N}$ (plus the $\frac{1}{2}$ contribution of the ground state). In the case of a single cash transfer of a quantity *s* at time $t = t_0$, the Hamiltonian becomes $\hat{H}(t) = s\delta(t - t_0)\hat{q}(t)$ where the delta function $\delta(t - t_0)$ has the value 1 at $t = t_0$ and 0 at other times.

The cash flow model treats the account as a black box which magically produces money at a fixed rate *r*. There are no inputs or outputs, which is why the Hamiltonian remains constant even as the nominal amount of money increases indefinitely. While such isolated systems do not exist in reality, the simple model – when coupled with the idea of quantum money creation – is instructive about how inflation occurs in something like a housing market. As emphasised in the book, money is created by private banks every time they issue a mortgage. If we assume mortgage lending continues at a steady rate, then the money supply will grow at some rate *r* (in Figure 3 of *Quantum Economics* the Canadian money supply grows at an annual rate of about 6.5 percent, so *r* = 0.065). If this money is then used to bid up the price of houses, then house price growth will track money supply growth, even if the real value of homes remains unchanged.

An important difference between cash and a security is that while money is a conserved quantity during transactions, a security once bought evolves into a superposition of states, each of different prices, with amplitudes specifying the probability of selling at that price. As an example, suppose that a particular investor initially has no shares in a particular company, and then acquires one share at time 0 for a price s_0.[15] By making a number of simplifying assumptions, and some rather involved computations, Schaden shows that the probability of

[15] The initial state $|M_0\rangle$ can be written $|M_0\rangle = \hat{b}^\dagger(s_0)|\widetilde{M}_0\rangle$ where $\hat{b}(s_0)|\widetilde{M}_0\rangle = 0$. Here the indices for other stocks and investors have been repressed for clarity, and $\widetilde{M}_0$ is a state where the investor has no shares in the company (which is why the annihilation operator yields 0). At time *T*, the state evolves to $|M_T\rangle = \hat{U}(t, t_0)|M_0\rangle$. The probability that the investor can sell the single share at a price s can be computed by looking at the product $\langle\widetilde{M}_T|\hat{b}(s)|M_T\rangle$, where $\widetilde{M}_T$ is again a state that is annihilated by $\hat{b}(s)$.

selling the stock a time T later for price s follows a lognormal distribution which depends on the expected return and volatility of the stock.[16] This is a well-known empirical result, that can be derived from standard stochastic approaches, so serves primarily as a consistency check. However it only holds for intermediate time scales of a month or more, and again assumes that the market is near equilibrium. The quantum approach helps to explain how this model breaks down at shorter time scales, or for assets which are infrequently traded.

To summarise this section, a market can be represented as a Hilbert space, in which the price of an asset is known precisely only at the time of a transaction. Ownership and context are important, so an asset purchased by one person at one price is distinct from the same asset purchased by another person at a different price. As in quantum cognition, the act of measuring an asset's price – in this case by buying or selling – has an effect on the price. By constructing an appropriate Hamiltonian equation, we can study the dynamics of market evolution. As in physics, the complexity of the system means that macro-level behaviour is often characterised by emergent properties that cannot be reduced to some lower level. Again this differs from the classical approach which assumes assets have a certain inherent value independent of context; money does not play an important role, other than as a metric; and calculations can be based on micro-foundations of individual utility optimisation.

Like quantum cognition, quantum finance has become a sizeable area of research, with many papers showing empirical results. If markets are assumed to be large and nearly efficient, then the results do generally approximate those produced by the classical approach. (Indeed, researchers have so far largely tended to respect classical assumptions such as efficiency, in an attempt to replicate known results.) However quantum effects become more important for markets that are thinly traded, and the quantum approach can also be used to describe markets driven by investor sentiment, where there is a significant degree of entanglement between market participants.

While quantum finance concentrates on the specialised case of financial markets, and is used for studying the properties of assets such as stocks or bonds, the same methodology can in principle be extended to describe markets in general, and form the basis of a mathematical

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[16] The formula is $P_T(s|s_0) = \frac{1}{s\sigma\sqrt{2\pi T}} \exp\left[-\frac{\left(\ln\left(\frac{s}{s_0}\right)-\mu T\right)^2}{2\sigma^2 T}\right]$ where μ is expected return and σ is volatility.

description of the quantum economy. Again, money has a special role as an asset with a fixed price, and the price of everything else is indeterminate until measured through monetary transactions.

6. Supply and demand

In neoclassical economics, supply and demand are treated as independent curves, whose intersection represents a stable equilibrium. In order to quantize the system, a first step is to represent it in dynamical terms, which means relaxing the condition of equilibrium. We can assume as a first approximation that the rate of price adjustment for some good is proportional to the amount of excess demand (though this may run into trouble once we consider the quantum dynamics of money).[17] Defining $\dot{x_d}$ to be the quantity demanded per unit time, and $\dot{x_s}$ to be the quantity supplied per unit time, the rate of change of price is then

$$\dot{p} = k(\dot{x_d} - \dot{x_s})$$

where k is a constant. (Note we are expressing supply and demand as rates in order to emphasise the dynamical nature of the system. For simplicity we are also using the price instead of the logarithm of price, which may be more appropriate since what counts is relative price.)

The dynamics of the system are then analogous to the situation sketched in the figure where two masses are joined by a spring, with spring constant k (we are assuming that the masses can pass through each other rather than colliding). Identifying the price as a force F, and the two masses m_s and m_d as the inertias of supply and demand respectively, gives a system similar to the harmonic oscillator, with the difference that there are now two masses which oscillate around their center of mass. The frequency of oscillation is found to be $\omega = \sqrt{k/m}$ where

$$m = \frac{m_d m_s}{m_d + m_s}.$$

[17] P. A. Samuelson, The stability of equilibrium: Comparative statics and dynamics. *Econometrica* 9(2), 97–120 (1941).

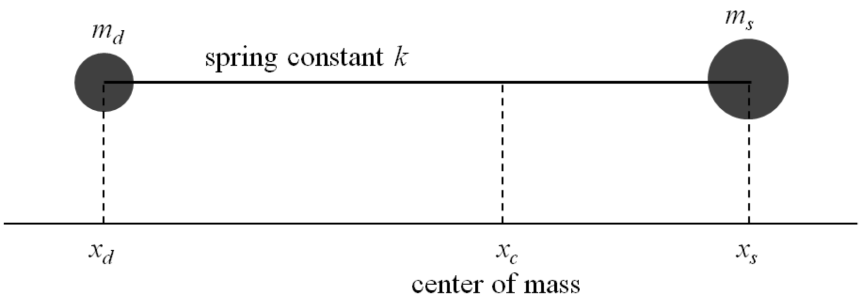

Figure 6.1. Supply and demand modelled as a spring system.

The amplitude of oscillation for demand and supply respectively are

$$A_d = \frac{m_s}{m_d + m_s} A$$

$$A_s = \frac{m_d}{m_d + m_s} A$$

where *A* is the maximum total extension. The system can therefore be treated as two harmonic oscillators with the same frequency but different amplitudes, and the equations quantized as before. The energy levels for demand correspond to the number of units demanded (by one or more customers), and energy levels for supply correspond to the number of units produced.

The mass terms represent inertias which resist change. A similar approach is taken in areas such as fluid dynamics, where the role of mass is played by viscosity: a particle in honey has more “mass” than one in water, which has more than one in air. The economy also behaves like a fluid because there are no isolated particles. Inertia is resistance to price change, and this “mass” can change if people expect it to change (e.g. they anticipate inflation). The economy therefore behaves like a non-Newtonian fluid. Perhaps the most unrealistic aspect of neoclassical economics is that it assumes the economy is at equilibrium, which means corrections occur instantaneously, as in the efficient market hypothesis. This is equivalent to saying that all mass terms are zero.

As an example, consider the case where m_s is extremely large relative to m_d, which corresponds to a situation where the seller produces at a constant rate. Then

$$m = \frac{m_d m_s}{m_d + m_s} \cong m_d$$

$$A_d = \frac{m_s}{m_d + m_s} A \cong A$$

$$A_s = \frac{m_d}{m_d + m_s} A \cong 0.$$

We are then back in the case of a single particle of mass m_d oscillating around a central point defined as the location of the larger mass. The equations for demand are

$$m\ddot{x}_d = -p + p_0$$
$$\dot{p} = k(\dot{x_d} - \dot{x_s}).$$

Here the equilibrium price p_0 can be written as $p_0 = \frac{\partial u(\dot{x}_d)}{\partial \dot{x}_d}$ where *u* is the “utility” gain per unit purchased, which we can assume to be constant for small perturbations.[18] (I will use the word “utility” to represent a general propensity or potentiality to make a purchase, which normally doesn’t coincide with actual usefulness.) Integrating the second equation gives $p - p_0 = kx$ where $x = x_d - x_s$ is the imbalance between supply and demand. The dynamics are then

$$q = m\dot{x}$$
$$\dot{q} = -kx$$

where q is the momentum. This describes an oscillator with $m\ddot{x} = -kx$. The Hamiltonian is

$$H = \frac{q^2}{2m} + \frac{1}{2} m\omega^2 x^2$$

where $\omega = \sqrt{k/m}$ is the frequency of oscillation. The first term represents the kinetic energy in terms of demand momentum, and the second term the potential energy in terms of deviation from balance with supply. (Units are discussed in the next section.)

To recap, we are assuming in this scenario that the supply rate is held constant, price is being dynamically adjusted in response to demand, and utility is a linear function of demand (a similar simplification can be obtained by assuming the inertial masses for supply and demand are identical, which may be appropriate for something like a stock market). To quantize the system, we follow the same procedure as for the quantum harmonic oscillator. Suppose the process is started in the ground state $|0\rangle$ meaning that market opinion is neutral. The state at

[18] A typical utility function is of the form $u(d) = a \exp\left(-\frac{d}{b}\right)$ which for small demand *d* can be approximated as $u(d) \cong a\left(1 - \frac{d}{b}\right)$, so the sensitivity to changes in demand is constant. Taking the derivative of u with respect to demand, and setting equal to the equilibrium price, gives a price-demand curve of the form $p(d) = \frac{a}{b} \exp\left(-\frac{d}{b}\right)$.

the next step, a time τ later, is determined from a unitary operator U acting on the initial state which accounts for changes in market sentiment. For example, suppose there is news that leads to a perceived shift in demand. As seen for the example of the displacement operator acting on the quantum harmonic oscillator, this leads to an increase in the system energy level and a corresponding reaction in both demand and price as in the classical model; however the results are now indeterminate until measured through a transaction. The resulting stochasticity can be important, especially when the number of transactions in a given period is small. The quantum model would take on an extra level of complexity if entanglement between classes of agents is considered, so that for example particular groups tend to align and react to news in the same way; and also if the equally entangling dynamics of financial transactions are taken into account.

The system can also be expressed as an oscillator in terms of relative price $r = p - p_0$ and price momentum z instead of demand, i.e.

$$z = m\dot{r}$$
$$\dot{z} = -kr.$$

The classical Hamiltonian in this case is

$$H = \frac{z^2}{2m} + \frac{1}{2}m\omega^2 r^2.$$

This form is commonly used in quantum finance as an expression of a stock's risk.[19] The first term captures the degree of price momentum, while the second reflects deviation from equilibrium. The mass m is seen as a measure of market capitalization, while ω reflects a characteristic oscillating frequency. (For a model of supply and demand in "price space" see the discussion paper *A Quantum Model of Supply and Demand*, available from https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3376652.)

An interesting difference between the classical and quantum versions is that in the former, if demand is perfectly matched to supply, then the market will always clear. In the quantum version there is an irreducible degree of uncertainty, with demand fluctuating above and

[19] For a similar result derived for a stock market, and comparisons with market data, see Meng, X. et al. (2015), "Quantum spatial-periodic harmonic model for daily price-limited stock markets", *Physica A: Statistical Mechanics and its Applications*, 438 (15), pp. 154-160.

below the equilibrium level.[20] One way to interpret this is as a measure of people's irrationality, however given that assets have no defined "intrinsic value" it is probably better seen as a reflection of investor uncertainty.

The quantum model has also been used to simulate stock return distributions over several years, based on the idea of a spring-like "reversion to the mean" force which attracts stock prices to a long-run equilibrium (though this assumes the existence of such an equilibrium in the first place).[21] The ground state of the oscillation is supplemented by higher energy levels which contribute the "fat tails" that characterise empirical distributions, and give a better fit to data than the Gaussian curve that is usually used.

The quantum approach is also compatible with Ising-type models from statistical mechanics which have been used to simulate stock market dynamics. In physics, the Ising model was initially developed to simulate ferromagnetic materials, where the magnetic dipole moments of atomic spins can be in one of two states ($+1$ or -1). When an external magnetic field is applied, interactions between atoms lead to phase transitions between a random state and ones in which spins are aligned.[22] The same idea can be applied to simulate contagion in the stock market. For example Gusev et al. (2015) created an empirically-fitted model where prices oscillate in a potential well which is determined in part by the propagation of news and opinions.[23] While they use a classical version of the Ising model, a quantum version would give similar results, though again with the feature of a ground state where fluctuations occur even in the absence of new information.

7. Production and consumption

A similar quantum approach can be applied to the production and consumption of goods. Suppose that we wish to model a firm which produces some good $x(t)$ at a rate $\dot{x}$, where the time dependence is suppressed for brevity. To understand the dynamics, it will be helpful to

[20] For a discussion in terms of the stock market, see *Quantum Economics: The New Science of Money*, pp. 112-116. See also: Zhang, C., and Huang, L. (2010), "A quantum model for the stock market", *Physica A: Statistical Mechanics and its Applications*, 389 (24), pp. 5769–75.
[21] K. Ahn, M. Y. Choi, B. Dai, S. Sohn and B. Yang (2017), 'Modeling stock return distributions with a quantum harmonic oscillator', *EPL* 120(3) , 38003.
[22] See discussion in *Quantum Economics*, p. 172.
[23] Gusev, M. et al. (2015), 'Predictable markets? A news-driven model of the stock market', *Algorithmic Finance*, 4 (1-2), pp. 5-51.

think in terms of dimensions. The rate is expressed in units per time, denoted LT^{-1}. In analogy with a physical system, we define the momentum as $q = m\dot{x}$ with units MLT^{-1} (mass times length over time), a force acting on the momentum as $s = \dot{q} = m\ddot{x}$ with units MLT^{-2}, and the work performed by the force over a distance x as $E = sx = m\ddot{x}x$ with units ML^2T^{-2}.

As an example based on the cash flow case described in Section 5 above, we could equate the savings account with a firm that magically produces money out of nothing. The good x is the amount of money, the production rate is $\dot{x} = rx$, the momentum is $q = m\dot{x} = mrx$, the accelerating force is $s = mr\dot{x}$, and the Hamiltonian is the energy

$$E = sx = mr\dot{x}x = rxq.$$

As earlier mentioned the energy of the system has units of currency E, which does seem an appropriate choice for economic energy. Since this must be equal to ML^2T^{-2} we conclude that the dimension M of mass is equivalent to $EL^{-2}T^2$. The momentum q therefore has units of ETL^{-1} while the force s has units EL^{-1} or price per unit. Because the system is blowing up in size (so becoming less dense) with no inputs of energy, the inertial mass term is not constant but decreases exponentially, with solution $m = m_0 e^{-2rt}$ where $m_0 = \frac{q_0}{rx_0}$. As in a nuclear reactor, the mass is being converted into another form of energy.

For modelling an actual productive company, rather than a cash-generating black box, one can use a similar framework in order to express the energy of the system, with the difference that we need to explicitly account for the costs of production through a cost function.[24] Consider first the case of a firm which produces a single item, with a profit function

$$g = p\dot{x} - c(\dot{x}).$$

This represents the money earned per unit time by selling units at a rate $\dot{x}$ and price p, minus the cost of production c. The units of g and c are therefore ET^{-1} (currency over time). The value of $\dot{x}$ which gives maximum profit can be found by setting

$$\frac{\partial g}{\partial \dot{x}} = p - \frac{\partial c(\dot{x})}{\partial \dot{x}} = 0.$$

In neoclassical economics, companies are generally assumed to be operating at this point, which doesn't allow for dynamics, or the fact that production won't usually be at an optimal level. However just as price acts as a force, so we can interpret this term $s = \frac{\partial g}{\partial \dot{x}}$ as a

[24] Dannenberg, A.A., Estola, M., and Dannenberg, A. (2017). A dynamic theory of economics: What are the market forces?

correcting force which includes costs and is directed towards optimum profitability. We then write $s = m_s \ddot{x}$ as before, where $\ddot{x}$ is the rate of change of production, and the mass term m_s now represents the inertia of the firm towards that change. If we assume that the inertial mass remains constant (which it need not) then the work performed by the force is

$$\Delta E = \int_0^t s dx = m_s \int_0^t \ddot{x} dx = m_s \int_{x_0}^{x_t} \dot{x} d\dot{x} = \frac{m_s}{2}\left(\dot{x}_t{}^2 - \dot{x}_0{}^2\right)$$

where $\dot{x_0}$ and $\dot{x_t}$ represent the initial and final production rates. We can therefore identify the terms of the form $\frac{1}{2} m_s \dot{x}^2$ as the economic equivalent of kinetic energy. The potential energy for a firm is the difference between this, and the maximum obtainable kinetic energy at the optimal level of productivity.

For a consumer, we can similarly write the "profitability" *w* from the purchase of a stream of goods *x* as

$$h = -p\dot{x} + u(\dot{x})$$

and the corresponding force towards that purchase as

$$\frac{\partial h}{\partial \dot{x}} = -p + \frac{\partial u(\dot{x})}{\partial \dot{x}}$$

where *u* represents the utility (see discussion of the demand case in the preceding section). The complete production/consumption system can then be coupled using the interaction between supply and demand, which relates price to sales. The dynamics are determined from the force equations for production, consumption, and supply and demand:

$$m_s \ddot{x}_s = p - \frac{\partial c(\dot{x}_s)}{\partial \dot{x}_s}$$

$$m_d \ddot{x}_d = -p + \frac{\partial u(\dot{x}_d)}{\partial \dot{x}_d}$$

$$\dot{p} = k(\dot{x_d} - \dot{x_s})$$

Again, for simplicity we are using price instead of its logarithm. Returning to the physical analogy of the spring, the system is therefore like the one in the Figure 6.1 above with two masses representing sellers and buyers, except that there are now two additional forces tugging on the masses that represent the producer's costs and the consumer's willingness to buy.

The equations are easily generalised to represent an arbitrary number of firms producing multiple goods. The main difference between the resulting model and the neoclassical version is that the latter sets the mass terms to zero in the equations for production and consumption, and assumes that the third equation representing the imbalance between supply and demand is equal to zero. As Dannenberg et al. (2017) demonstrate, the act of changing from the static neoclassical framework to a dynamical framework is in itself sufficient to reproduce some aspects of phenomena such as economic crises (see their paper for simulations). The equations can also be quantized as before, with the difference that there are now extra force terms which account for costs and consumer desires. The quantum model could simulate entanglement through the introduction of coupling terms.[25] The inclusion of money and debt would also allow one to model the effects of credit, which is what permits companies (e.g. Tesla) to operate for long periods on borrowed funds. The complexity of the resulting model would probably limits its application to anything but highly simplified situations; however as discussed further below targeted models could be useful for exploring a range of economic phenomena.

As a final note, one can also write the equations in the form

$$\dot{v}_s = \frac{1}{m_s}\left(p - \frac{\partial c(v_s)}{\partial v_s}\right)$$

$$\dot{v}_d = \frac{1}{m_d}\left(-p + \frac{\partial u(v_d)}{\partial v_d}\right)$$

$$\dot{p} = k(v_d - v_s)$$

where $v_d = \dot{x_d}$ and $v_s = \dot{x_s}$. These compare directly with the equations used in systems biology to describe the interactions of three chemical species. In biology, stochastic effects occur because key reactions, such as the production of mRNA, often involve only a small number of molecules, so reactions are described using the same type of Poisson process.[26] Results for those models can therefore be carried over to economics. For example biological systems sometimes include feedback loops that limit stochastic fluctuations[27] and it would be interesting to compare their function with human attempts to smooth economic fluctuations.

8. Entanglement

[25] Kim, YS and Noz, ME (2005). Coupled oscillators, entangled oscillators, and Lorentz-covariant harmonic oscillators. *Journal of Optics B: Quantum and Semiclassical Optics* 7(12): S458.

[26] Ramsey, S., Orrell, D., & Bolouri, H. (2005), 'Dizzy: stochastic simulation of large-scale genetic regulatory networks', *Journal of bioinformatics and computational biology* 3 (02), 415-436.

[27] Ramsey, S., et al. (2006), 'Dual feedback loops in the GAL regulon suppress cellular heterogeneity in yeast', Nature genetics 38(9): 1082.

As discussed in the book, a key advantage of the quantum approach in economics – but one which to my knowledge has not previously been addressed by researchers in quantum finance – is that it provides a natural framework for thinking about financial entanglement through loans and derivatives.

To first motivate the discussion, consider the physical example of a pair of entangled electrons, denoted 1 and 2, each of which has spin ½ when measured along a particular axis, but in opposite directions. The spin part of their wave function can be written as a superposition of two states:

$$|S\rangle = \frac{1}{\sqrt{2}}\,|1\uparrow\rangle|2\downarrow\rangle - \frac{1}{\sqrt{2}}|1\downarrow\rangle|2\uparrow\rangle$$

where the arrow indicates the direction of spin of each electron.

The wave function tells us nothing about the direction of spin for either electron, only that they are opposite, so the total spin is zero. Now, suppose that we measure the spin for electron 1. We would expect an equal chance of getting a positive or negative result. If it is the former, then the system must have collapsed to an eigenstate with positive eigenvalue, so is of the form

$$|S\rangle = |1\uparrow\rangle|2\downarrow\rangle$$

A measurement of particle 2 can now yield only a negative result. The reason is that the wave function describes the system, including both particles, so a measurement on one is equivalent to a measurement on the system as a whole.

The financial version of entanglement can be expressed using a similar formalism. Instead of two entangled electrons, consider two people entangled by a loan contract; and instead of spin direction, we will use loan status (i.e. "default" or "no default"). As in quantum cognition, the debtor is modelled as initially being in a superposition of two states, with a decision acting as a measurement event. The loan status can therefore be expressed by a wave function of the form:

$$|S\rangle = \alpha\,|1\uparrow\rangle|2\downarrow\rangle - \beta|1\downarrow\rangle|2\uparrow\rangle$$

Here α^2 and β^2 add to 1, and give the probability of default $|1\uparrow\rangle|2\downarrow\rangle$ and no default $|1\downarrow\rangle|2\uparrow\rangle$ respectively, so reflect the debtor's propensity to default at a particular moment. If the debtor decides to default on the loan, that acts as a measurement on the system as a whole. At any time after that, if the creditor decides to assess the state of the loan, the result can only indicate default. The two parties are thus entangled.

Of course, systems can be correlated without any need to invoke quantum effects.[28] However the key point is that we are treating the debtor's state regarding the loan as being in a superposition of the two states "default" and "no default". The state of the loan is therefore indeterminate (we don't know whether the debtor will default) yet still correlated, which is the essence of entanglement.

Another possible objection is that, after one of a pair of entangled particles has been measured, the second doesn't need to check with the first to find out what its state is; while with a loan the creditor does. However the wave function equation applies to the loan agreement, which is an abstract thing that encompasses both parties. So from the point of view of that wave function (which again is what we are modelling) the state does change instantaneously; it is only measurements that take time. The difference between the physics version, and the financial version, then reduces to a question of the nature and reality of such wave functions, which would depend on one's interpretation of quantum theory, and is a topic of debate for both physicists and social scientists.[29] But from a mathematical modelling perspective the two are the same.

One feature of the system is that, unlike for electrons, there is now only one axis of measurement. This means that the behaviour of a loan agreement is much less subtle than the physical version (though some social scientists do argue for rich versions of mental entanglement based on physical principles); and also that it is not possible to reproduce Bell-type experiments, where entanglement is tested by changing the orientation of the axis. However Bell's experiments do not define entanglement, but were devised as a way to tease out entanglement for systems that cannot be queried more directly. For loans, the entanglement is encoded by the terms of the agreement. Again, the equation applies only to the loan agreement, so default may for example be followed by a complex negotiation, but the same is true in a physical system where other forces can also come into play.

[28] For example, suppose I have two beads, one red and one blue, and I give one to a friend without looking. Then if I check and find that I have the red one, I know that my friend has the blue one.
[29] A widely discussed example is whether Xantippe, the wife of Socrates, became a widow the instant her husband was forced to commit suicide, or only when she found out later. See: Wendt, A. (2015), *Quantum Mind and Social Science: Unifying Physical and Social Ontology* (Cambridge: Cambridge University Press), p. 194.

Since most money is produced through private bank lending, and the financial system is dominated by complex derivatives contracts, financial entanglement is a tremendously important part of the economy, yet one which has been largely neglected in mainstream models, precisely because they are based on a classical atomistic paradigm. A number of techniques are currently being developed to simulate collective decision-making using a quantum approach, and these could be used to model phenomena such as mass defaults, or the impact of collective behaviour on the generation of credit in an economy.[30]

9. Summary

The quantum approach has successfully been used to model the economy at both the level of individuals (in quantum cognition) and the level of markets (quantum finance). In either case, the state of the system is represented using a Hilbert space. Measurement procedures such as decisions and transactions take precedence over internal states such as known preferences or inherent values. The quantum approach therefore differs fundamentally from the classical one, and can be extended to offer an alternative model of the economy in general.

While the aim of this document is only to give an idea of how quantum techniques can be applied to the economy, the literature in this area is quite large and different researchers take different approaches. As shown by empirical results in quantum cognition, the quantum approach appears to be a natural fit for modelling human decision-making. And while quantum finance has not been widely adopted by the quantitative finance community, some traders have adopted the quantum methodology to understand and predict for example the behaviour of illiquid assets.

From the larger perspective of quantum economics, a main advantage of the quantum approach is that it naturally incorporates the dualistic properties of money. In classical economics, price is essentially equated with value (with allowances for “market failures”). In quantum mechanics, prices are seen as emerging from monetary transactions. One consequence is to sever the direct link between price and value. Another is to concentrate the modeller’s attention on the entangling properties of money.

[30] One researcher investigating quantum models of collective decision making is Michael Schnabel: https://harris.uchicago.edu/directory/michael-schnabel

A natural extension of the market models considered above, and an interesting longer-term research project, would be a quantum agent-based model of something like a housing market. Following the approach used to model the propensity to buy or sell individual stocks (Section 6), each house could be considered as a separate single-asset market. Buyers and sellers would be entangled to a degree with each other, and to the news flow which could be modelled as a quantum variable, and to the financial markets through loans. Such a model could simulate the kind of market contagion seen in housing markets, such as “fear of missing out” when prices are rising. It could also include the process of money creation through private lending, which grows the money supply and leads to asset price inflation.

To summarise, the quantum approach offers a natural framework for modelling key economic properties such as indeterminacy and entanglement. It also explicitly accounts for stochastic dynamic effects, of the sort that are regularly studied in areas such as systems biology, but have played a much smaller role in economics (apart from finance). While many people with a background in physics will be familiar with the quantum approach, and can easily apply methods from e.g. statistical mechanics to derive results, those trained in a classical approach may at first find it awkward or overly elaborate. However one of the main lessons of quantum economics is that, just because the economy *emerges* from quantum effects, this does not imply that quantum models are always obligatory. The complex behaviour of water, which ultimately arises from quantum properties, may drive the weather system but is not part of weather models; and similarly it is possible to simulate the flow of money in a way that respects its complex emergent properties without needing to go down to the quantum level. The quantum approach can also be used to rule out certain modelling approaches, including Dynamic Stochastic General Equilibrium models (the so-called workhorses of macroeconomics), which rely on classical assumptions such as equilibrium.

For more background and further reading, see davidorrell.com/quantumresources.html
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