

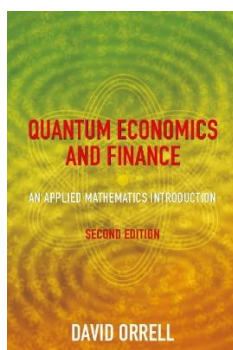
Introduction to the mathematics of quantum economics

by David Orrell

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Note: This article is no longer maintained. For an updated treatment and latest findings see the book:

Orrell D (2020) *Quantum Economics and Finance: An Applied Mathematics Introduction*. New York: Panda Ohana.



http://www.postpythagorean.com/book_quantumeconomicsfinance.html

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(1) Use mathematics as a shorthand language, rather than an engine of inquiry. (2) Keep to them till you have done. (3) Translate into English. (4) Then illustrate by examples that are important in real life. (5) Burn the mathematics.

Alfred Marshall, 1906²

Too large a proportion of recent “mathematical” economics are merely concoctions, as imprecise as the initial assumptions they rest on, which allow the author to lose sight of the complexities and interdependencies of the real world in a maze of pretentious and unhelpful symbols.

John Maynard Keynes, 1936³

Nature isn't classical, dammit, and if you want to make a simulation of nature, you'd better make it quantum mechanical.

Richard Feynman, 1981⁴

1. Introduction

This document gives a technical introduction to some of the mathematics used in quantum economics, and is intended as a supplement for the book *Quantum Economics: The New Science of Money*. As the quotes above point out, economics is not the same as a mathematical proof, and the key ideas of quantum economics, such as the quantum theory of money and value, do not rely on equations. However the quantum formalism *is* mathematical, so to fully exploit its ideas some mathematics is useful (even if it is burned afterwards). The aim here is to sketch out the way in which the economy can be represented mathematically using the quantum formalism, show the advantages over the classical approach, and clarify (at least for those with some knowledge of basic matrix algebra) what it means to say that the economy can be treated as a quantum system in its own right.

The quantum approach to economics is inspired by the empirical fact that the monetary system shows quantum properties such as discreteness, indeterminacy, entanglement, and so on. To borrow Feynman's expression, a simulation had therefore better be quantum

² “(6) If you can't succeed in 4, burn 3. This last I do often.” Letter to A.L. Bowley, 27 February 1906.

³ From *The General Theory of Employment, Interest and Money*.

⁴ From a 1981 talk “Simulating Physics with Computers” on the idea of a quantum computer.

mechanical too, in the sense that it reflects these properties (even if it doesn't directly use a quantum formalism). The point is therefore not that the quantum approach will be the best technique to model every aspect of the economy, but rather that the economy has quantum properties which may need to be taken into account (explicitly or implicitly) depending on the context.

As an example from physics, weather forecasters do not base their models on quantum mechanics, but they do base them on the complex properties of water, which emerge from quantum mechanics. So in this case the main lesson is that the quantum properties of water molecules lead to highly complex emergent properties at the global level, which resist reduction to a lower level. In economics, one might therefore conclude that economic behaviour should be modelled at the appropriate level, so quantum properties are not directly relevant. On the other hand, in physics a technology such as an atom bomb scales quantum properties up to the macro level. Similarly, money is a designed technology, and its properties sometimes scale up to affect the economy as a whole, for example through phenomena such as money creation by private banks.

Models are ultimately justified by their success at explaining and predicting data. While the focus here is on presenting the basic tools of the theory, and showing how they relate to the nature of economic transactions, rather than on specific results, it should be noted that the areas of quantum cognition and quantum finance are heavily empirical, basing their results on experimental data for the former, and market data for the latter. The broader area of quantum economics – dealing as it does with emergent properties of a complex system – incorporates in addition a variety of complexity-based techniques, from agent-based models to systems dynamics, which have also been empirically tested (an exception is quantum agent-based models, which to my knowledge have yet to be developed for economics). The most obvious empirical argument for the quantum approach, though, is simply the nature of money, which is designed to have quantum properties.⁵ For details, please see the book, and the references therein.

An outline is as follows. Section 2 introduces the idea of the Hilbert space, and shows how quantum probability differs from its classical version using the example of human cognition.

⁵ See: Orrell, D. (2018), 'Quantum Economics', *Economic Thought*, 7 (2).

Section 3 shows where classical approaches to cognition of the sort used in behavioural economics break down, and Section 4 illustrates the quantum approach using two examples. Section 5 discusses the quantization procedure for a dynamic system, and applies it to the paradigmatic example of the quantum harmonic oscillator. Section 6 uses the same ideas to develop a quantum model of a market, where shares and cash now take the place of bosons. Section 7 explores the quantum representation of supply and demand. Section 8 extends this dynamical analysis to production and consumption, and points towards how one could construct quantum models for more general applications. Section 9 discusses the concept of entanglement, and Section 10 summarises the main conclusions.

2. Some basics

Perhaps the most basic mathematical tool in quantum theory is the concept of the Hilbert space, which is named for the German mathematician David Hilbert (1862-1943). It was developed as an abstract mathematical object in the first decade of the twentieth century, and was later adopted by researchers in quantum physics. Social scientists are now following their lead by applying it to problems in areas such as decision-making and finance, as seen below.⁶

A Hilbert space H is a type of vector space whose elements, denoted $|u\rangle$, have coefficients that can be complex numbers. The dual state $\langle u|$ is the complex conjugate of the transpose of $|u\rangle$. The inner product between two elements $|u\rangle$ and $|v\rangle$ is denoted $\langle u|v\rangle$, and is analogous to the dot product in a normal vector space, with the difference that the result can again be complex. The outer product is denoted $|u\rangle\langle v|$, and is like multiplying a column vector by a row vector, which yields a matrix. The magnitude of an element $|u\rangle$ is given by $\sqrt{\langle u|u\rangle}$, and two elements are orthogonal if $\langle u|v\rangle = \langle v|u\rangle = 0$. The Hilbert space can therefore be viewed as a generalisation of Euclidean space, with the difference that there can be an infinite number of dimensions (though conditions apply), the basis need not be simple column vectors, and coefficients can be complex.

⁶ Some researchers in cognitive science prefer to treat the Hilbert space as just a tool, and see the word “quantum” as a distraction. Irving Fisher, in his 1892 book *Mathematical Investigations in the Theory of Value and Prices*, had a similar problem with the word “utility” which he described as “the heritage of Bentham and his theory of pleasures and pains. For us his *word* is the more acceptable, the less it is entangled with his *theory*” (p. 23). Personally I think it would be a little forced to ignore the theory’s connections with physical reality.

An operator \hat{A} is a map which sends one element $|u\rangle$ of H to another element $\hat{A}|u\rangle$ of H . For example, the projection operator is defined as $\hat{Q}_u = |u\rangle\langle u|$, and $\hat{Q}_u|v\rangle = |u\rangle\langle u|v\rangle$ gives the projection of v onto u . Operators \hat{A} and \hat{B} do not generally commute, so $\hat{A}\hat{B} \neq \hat{B}\hat{A}$. A state $|u\rangle$ is an eigenvector of \hat{A} if $\hat{A}|u\rangle = \lambda|u\rangle$ where λ is the associated eigenvalue. For example $\hat{Q}_u|u\rangle = |u\rangle\langle u|u\rangle = |u\rangle$, so $|u\rangle$ is an eigenvector of \hat{Q}_u with eigenvalue $\lambda = \langle u|u\rangle$. The expectation value of a linear operator \hat{A} in the state $|u\rangle$ is given by $\langle u|\hat{A}|u\rangle$, i.e. the scalar product of $\langle u|$ with $\hat{A}|u\rangle$.

The quantum state of the system is represented mathematically by a ray in a Hilbert space over the complex numbers. A key feature of quantum theory is that observables such as a particle's position or momentum are represented by Hermitian operators \hat{O} , which have real eigenvalues, and eigenvectors that form an orthonormal basis of the state space.⁷ Any quantum state $|S\rangle$ can therefore be represented by a superposition of these eigenstates. A measurement procedure causes the state to collapse to one of the eigenstates, and returns the corresponding eigenvalue as the measurement.

For example, the position operator \hat{X} has eigenvalue x and corresponding eigenvector $|x\rangle$. The eigenvalues correspond to the possible position vectors of the system. For a projection operator, the eigenvector is the vector being projected onto, and the eigenvalue is 1. If the operator has a differential in it, like the momentum operator, then as seen below the eigenvalue equation becomes a differential equation.

For the 2-D case, suppose that the eigenvectors of the observable \hat{O} are u_1 and u_2 , with eigenvalues λ_1 and λ_2 . If the eigenvectors form an orthonormal basis, then the system state can be written as the superposition $|S\rangle = c_1u_1 + c_2u_2$. If the state is normalised, so that $|c_1|^2 + |c_2|^2 = 1$, then the Born rule states that the probability of a measurement returning the eigenvalue λ_1 is equal to $|c_1|^2$, and the probability of λ_2 is $|c_2|^2$.

To see the difference between the classical and quantum approaches, in the context of human cognition, suppose that a person has a choice between a certain number of possible options.

In classical probability theory, each choice u would be treated as a subset of the set U

⁷ A Hermitian operator is one which equals its Hermitian conjugate, which for a matrix operator is defined as the complex conjugate of the transpose, so $A = A^\dagger \equiv (A^T)^*$.

consisting of all choices. A person's cognitive state is represented by a function p with the probability of choosing X given by $p(u)$. As a simple example, U could consist of two choices u and v , with respective probabilities $p(u)$ and $p(v)$, that satisfy $p(u) + p(v) = 1$.

In quantum cognition, a choice in response to a particular question is treated instead as an element (e.g. vector) $|u\rangle$ of a Hilbert space H , and a person's cognitive state is represented by an element $|S\rangle$, both of length 1. (The state $|S\rangle$ is sometimes called a wave function, although here it is static rather than time-varying.) Here the associated operator \hat{Q}_u is the one that projects vectors onto $|u\rangle$, so $|u\rangle$ is the eigenvector (as seen later, it is possible to work either with the vector $|u\rangle$ or with the corresponding projection operator). The probability of the answer to the question being $|u\rangle$ is then given by the magnitude of the projection squared, which is $P(u) = |\langle u|S\rangle|^2$.

As a simple example, the two axes in the figure below represent decisions of Yes or No to some question, while a person's state is represented by the grey line at an angle α to the No axis. The probability of deciding Yes is given by the square of the projection onto the Yes axis, which equals $\sin^2 \alpha$.

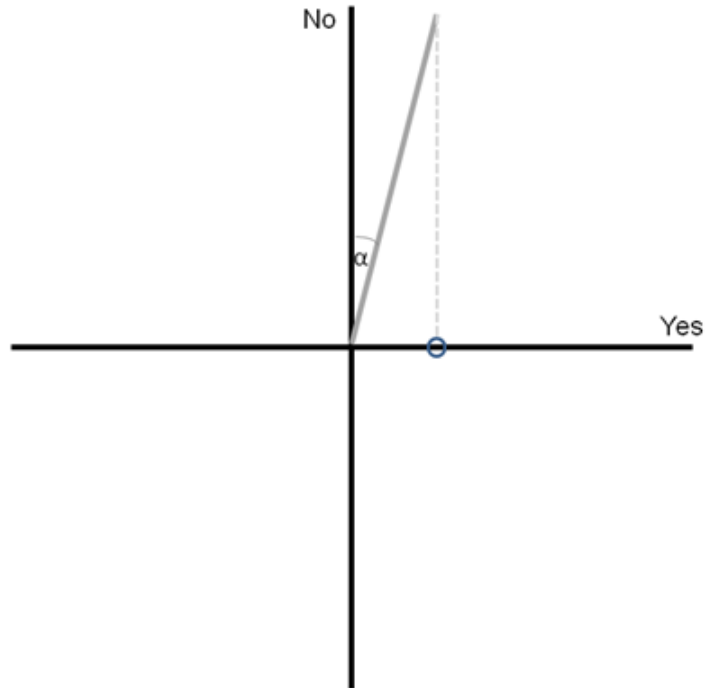


Figure 2.1. Axes show decision states which correspond to eigenvectors, grey line shows a person's state $|S\rangle$. The probability of deciding Yes is found by projecting onto the Yes axis and taking the square, which gives $\sin^2 \alpha$.

This shift, from sets of elements to geometric projections, allows for more complicated probabilistic effects such as non-commutativity and interference, which as seen in the next section are characteristic of human cognition.

3. Human cognition

Neoclassical economics is based on the theory of expected utility, which was first codified by the Hungarian mathematician John Von Neumann and the economist Oskar Morgenstern in their 1944 book *Theory of Games and Economic Behaviour*. The aim was “to find the mathematically complete principles which define ‘rational behavior’ for the participants in a social economy, and to derive from them the general characteristics of that behavior.” They arrived at a list of four principles or axioms.

Suppose an agent is faced with two games or lotteries A and B with different potential payoffs. The Completeness axiom then assumes that the agent has well-defined preferences and can always choose between the two alternatives. The Transitivity axiom assumes that if the agent prefers A over B, and B over C, then they prefer A over C. The Independence axiom assumes that, if the agent prefers A over B, then introducing an unrelated lottery C does not change that preference. Finally, the Continuity axiom assumes that if the agent prefers A over B, and B over C, then there should be some mix of the most-favoured A and the least-favoured C which is equally attractive as B. If the agent meets these four axioms, then their preferences can be modelled using a so-called utility function, and they are officially rational.

The expected utility for each lottery is defined as the sum of utilities of the possible outcomes, weighted by the probability of each outcome. Suppose for example a lottery A has two possible payoffs: an amount a_1 with probability $p(a_1)$, and an amount a_2 with probability $p(a_2)$. The expected utility is then

$$U(A) = p(a_1)u(a_1) + p(a_2)u(a_2) = p(a_1)a_1 + p(a_2)a_2$$

since the utility of a payout is here just the payout. A lottery B is preferred if its expected utility satisfies $U(B) > U(A)$.

While expected utility theory still forms the basis for most models in economics, since the 1970s behavioural psychologists and economists have shown that the theory doesn't capture a variety of cognitive phenomena. One of the first attempts to modify expected utility theory was the prospect theory of Kahneman and Tversky, published in their 1979 paper "Prospect Theory: An Analysis of Decision under Risk". This modified expected utility theory in two ways. The first was to say that what counts is not final amounts, but wins or losses relative to some reference point. The second was to say that outcomes are weighted by a nonlinear uncertainty weighting function, rather than probability itself. These two main findings of prospect theory are illustrated in Figures 3.1 and 3.2.

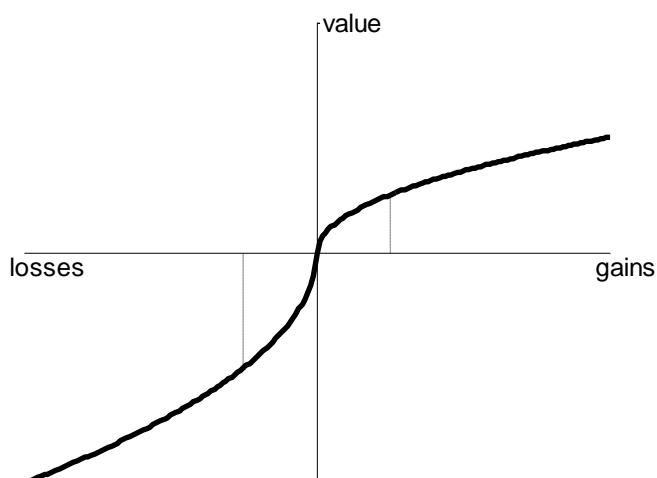


Figure 3.1. Plot of a value function, showing the psychological value of an event as a function of monetary gains or losses. The centre represents the reference level, according to which gains or losses are experienced. The function saturates for large gains or losses, and is also asymmetric around the origin because a loss of a certain amount is felt more keenly than a similar gain (dotted lines).

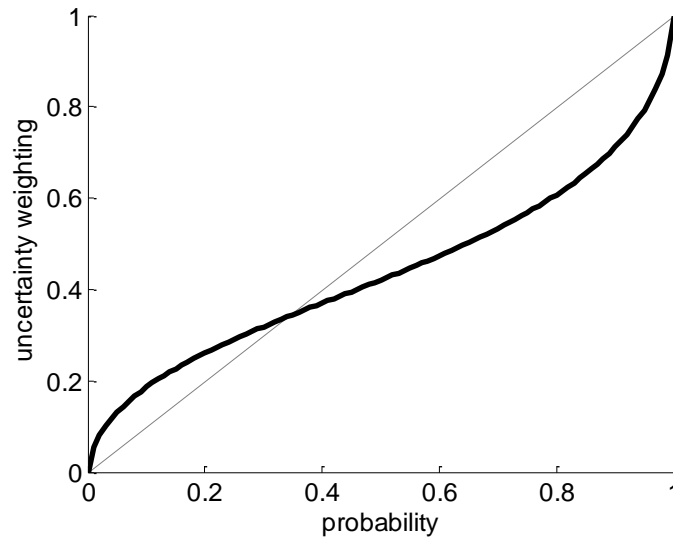


Figure 3.2. Plot of an uncertainty weighting function. In expected utility theory, the uncertainty weighting of an event equals its probability (dashed line). In prospect theory, the curve is concave near 0 and convex near 1 (solid line). The curve will be different for losses and gains, but similar in shape.

Following (Tversky & Kahneman, 1992) the value function $v(x)$ and uncertainty weighting function $w(p)$ were generated using the following equations:

$$v(x) = -2(-x)^{0.5} \text{ for } x < 0$$

$$v(x) = x^{0.5} \text{ for } x \geq 0$$

and

$$w(p) = \frac{p^\gamma}{(p^\gamma + (1 - p)^\gamma)^{1/\gamma}}$$

where $\gamma = 0.61$.

Together, these figures summarise many of the key cognitive phenomena which form the core of behavioural economics. For example, losses and gains are felt relative to some reference point, which will depend on the context. This point is represented by the zero of the horizontal axis for the value curve in Figure 3.1. Most people are loss averse, in the sense that a loss of a certain amount is roughly twice as painful as a gain of the same amount is pleasurable. This is why the value curve is asymmetrical around the origin, with a steeper slope for losses. The shape also reflects the finding, which goes back to the eighteenth century mathematician Daniel Bernoulli, that the effect of losses or gains tends to saturate at larger amounts.

Experiments also show that we don't weight outcomes exactly by their probabilities. In particular, we are more sensitive to a change in probability from 0 to 0.1, or from 0.9 to 1, than we are to a change from 0.4 to 0.5, which is why the uncertainty function in Figure 3.2 is concave near 0 and convex near 1. This is why for example we give too much weight to reports of low-probability events like terrorist attacks or other unlikely disasters.

To summarise, the difference between prospect theory and expected utility theory is that instead of writing

$$U(A) = p(a_1)a_1 + p(a_2)a_2$$

we write

$$U(A) = w(a_1)v(a_1) + w(a_2)v(a_1)$$

where v is the value function and w is the uncertainty weighting function. Prospect theory can therefore be viewed as a modified version of expected utility theory, where linear relationships are replaced by nonlinear curves. As an example of how it is applied, consider the following two games, which give an example of the Allais paradox first described by the French economist Maurice Allais in 1952.

Game A: choose between

- a₁: \$40 with probability 80%
- a₂: \$30 with probability 100%

Game B: choose between

- b₁: \$40 with probability 20%
- b₂: \$30 with probability 25%

According to expected utility theory, we have

$$U(a_1) = p(a_1)a_1 = 0.80 * 40 = 32$$

$$U(a_2) = p(a_2)a_2 = 1.00 * 30 = 30$$

for Game A, and

$$U(b_1) = p(b_1)b_1 = 0.20 * 40 = 8$$

$$U(b_2) = p(b_2)b_2 = 0.25 * 30 = 7.50$$

for Game B. In either game (or prospect, as it is called) the first option offers a slightly better expected utility. However in practice people usually choose the first option for Game B, but the second option for Game A. The reason is that Game A includes a “sure thing” option, which is more attractive. However this implied their utility theory was not consistent, which violated the axioms of expected utility theory.

In prospect theory, using the above versions of the value and uncertainty weighting functions, these calculations become

$$U(a_1) = w(0.80)v(40) = 0.61 * 6.32 = 3.86$$

$$U(a_2) = w(1.00)v(30) = 1.00 * 5.48 = 5.48$$

for Game A, and

$$U(b_1) = w(0.20)v(40) = 0.26 * 6.32 = 1.64$$

$$U(b_2) = w(0.25)v(30) = 0.29 * 5.48 = 1.59$$

for Game B. The most attractive options are now a_2 and b_1 , in agreement with experiments. The reason is that the sure-thing option is overweighted in terms of psychological value.

While prospect theory does address many of our cognitive quirks, there are a number of others which require separate attention. An example is the Ellsberg paradox. This involves an urn containing 90 balls, of which 30 are red and 60 are either black or yellow. You are given the choice between two gambles.

In Game A, you bet on either red or black.

In Game B, you bet on red or yellow, or black or yellow.

Which would you prefer? In each game the chances of drawing a red, black or yellow ball are one in 3. The only difference between the games is that in Game B, each side of the bet includes yellow. So if you prefer red in Game A, then you should prefer “red or yellow” in Game B. However most people see it differently – they don’t look at the colour of the ball, but at the uncertainty. In Game A, the number of red balls is known to be 30, but the number of black balls is uncertain. They therefore choose red in Game A. In Game B, the number of yellow balls is uncertain, however the sum of black and yellow balls is known to be 60. They therefore choose to bet on “black or yellow”, since again that is the option with less uncertainty.

This inconsistency contradicts expected utility theory, however it also eludes prospect theory, for the simple reason that the probabilities are unknown, so it is impossible to adjust them with the uncertainty weighting function. The paradox could only be explained by introducing a new and different kind of ad hoc weighting function, which accounted for uncertainty aversion.⁸

In fact it turns out that there are many other cognitive phenomena which cannot be captured in a straightforward way using classical theory. These include the so-called conjunction and disjunction effects, the order effect, and preference reversal. The thing they have in common is that in all cases, the context and the measurement procedure affects the answer, as with a quantum measurement. In the Ellsberg paradox for example the two options are formally identical, with the only difference being the details of the scenario. The uncertainty about the number of black or yellow balls creates a kind of mental interference pattern that affects judgment. The next section shows how the quantum approach can be used to resolve such paradoxes.

4. Quantum cognition

To return to the example of the Allais effect, we first note that the problem can be expressed in matrix form by setting the payoff matrix for Game A to

$$U = \begin{pmatrix} u_1 & 0 & 0 \\ 0 & u_2 & 0 \\ 0 & 0 & u_3 \end{pmatrix} = \begin{pmatrix} 40 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and the probabilities as vectors $\psi_1 = (\sqrt{0.8} \quad 0 \quad \sqrt{0.2})$ and $\psi_2 = (0 \quad 1 \quad 0)$, where the third component refers to the chance of winning nothing. The expected utility is then given by

$$O(\psi) = \psi U \psi^T$$

so

$$O(\psi_{a1}) = (\sqrt{0.8} \quad 0 \quad \sqrt{0.2}) \begin{pmatrix} 40 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{0.8} \\ 0 \\ \sqrt{0.2} \end{pmatrix} = 32$$

⁸ This involves something known as “non-additive probabilities” or “capacities” (Halpern 2003).

$$O(\psi_{a2}) = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 40 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 30$$

$$O(\psi_{b1}) = \begin{pmatrix} \sqrt{0.2} & 0 & \sqrt{0.8} \end{pmatrix} \begin{pmatrix} 40 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{0.2} \\ 0 \\ \sqrt{0.8} \end{pmatrix} = 8$$

$$O(\psi_{b2}) = \begin{pmatrix} 0 & \sqrt{0.25} & \sqrt{0.75} \end{pmatrix} \begin{pmatrix} 40 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \sqrt{0.25} \\ \sqrt{0.75} \end{pmatrix} = 7.50$$

as before.

We can also use the spectral decomposition of U to write

$$O(\psi) = \psi U \psi^T = \psi \left(\sum_i u_i V_i V_i^T \right) \psi^T = \sum_i u_i |\psi V_i|^2$$

where V_i is the column eigenvector of O corresponding to the eigenvalue u_i , and $V_i V_i^T$ is the projection operator that projects onto that eigenvector. For the classical case, where O is diagonal, the matrix V of eigenvectors is the identity matrix, so the equation reduces to a weighted expected utility function, where the payoffs u_i are weighted by the probabilities $|\psi V_i|^2 = \psi_i^2$. In this case for example we can write

$$O(\psi) = 40\psi \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \psi^T + 30\psi \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \psi^T + 0\psi \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \psi^T.$$

In the quantum framework, the matrix U corresponds to an observable, which again is a Hermitian operator with real eigenvalues.⁹ The expected utility $O(\psi)$ is the expected value of this observable when the system is in the state

$$|\psi\rangle = \sum_i \langle V_i | \psi \rangle \cdot |V_i\rangle$$

where the weights $\langle V_i | \psi \rangle^2$ correspond to the terms $|\psi V_i|^2$ in the matrix version, and represent the subjective weight assigned to the payoff u_i .¹⁰ In other words, we decompose the state ψ of the decision-maker into the basis of eigenvectors V_i , each of which corresponds to a particular outcome, and use the result to obtain the observable $O(\psi)$.

⁹ For an Hermitian matrix, all eigenvalues are real, and eigenvectors corresponding to distinct eigenvalues are orthogonal.

¹⁰ Busemeyer and Bruza, 2012: 259.

In the classical case, the probabilities ψ refer to the objective state of the game, and the weights combine these probabilities with the payoffs V_i . The problem is that there is no penalty given to reflect the fear of winning nothing, which is known to influence decisions. Following La Mura's (2009) theory of "projected expected utility", this can be done in the quantum framework by choosing the observable U to be a Hermitian operator where the off-diagonal terms are non-zero.¹¹ If this is the case, then the weights $\langle V_i | \psi \rangle^2$ will not just weight an outcome by its probability of occurring, but can include other factors, such as the fact that an outcome occurring means that another outcome did not occur. For example, a small degree of aversion to the risk of winning nothing (the third component) can be modelled by setting the observable to

$$U = \begin{pmatrix} 40 & 0 & -5 \\ 0 & 30 & -5 \\ -5 & -5 & 0 \end{pmatrix}$$

which gives

$$O(\psi_{a1}) = (\sqrt{0.8} \quad 0 \quad \sqrt{0.2}) \begin{pmatrix} 40 & 0 & -5 \\ 0 & 30 & -5 \\ -5 & -5 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{0.8} \\ 0 \\ \sqrt{0.2} \end{pmatrix} = 28$$

and similarly $O(\psi_{a2}) = 30$, $O(\psi_{b1}) = 4$, and $O(\psi_{b2}) = 3.17$, so now options a_2 and b_1 are selected.

The eigenvalues of U are now 40.65, 30.75, and -1.40, which shows that the third outcome of winning nothing is being given a negative weight. The projection operators, which are orthogonal to each other, are also rotated by the matrix whose columns are the new eigenvectors. So instead of

$$V_1 V_1^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

in the classical version, we now have

$$V_1 V_1^T = \begin{pmatrix} 0.9796 & 0.0600 & -0.1279 \\ 0.0600 & 0.0037 & -0.0078 \\ -0.1279 & -0.0078 & 0.0167 \end{pmatrix}.$$

To summarise, a diagonal utility matrix means that prospects are being evaluated by projecting onto a basis consisting of objective outcomes with known utilities. In practice however a person will evaluate a prospect by projecting onto a subjective basis which will

¹¹ La Mura, Projective Expected Utility. <https://arxiv.org/pdf/0802.3300.pdf>

not align perfectly with the classical version, and will assign a negative weight to undesired outcomes.

The same approach can be applied to the Ellsberg paradox.¹² If we assume that the prize for winning the gamble is \$100, then the utility matrix is

$$U = \begin{pmatrix} 100 & 0 \\ 0 & 0 \end{pmatrix}.$$

The risk vector $c_1 = (\sqrt{1/3} \quad \sqrt{2/3})$ corresponds to the game of betting on red (which has a 1/3 probability of success). The risk vector $c_2 = (\sqrt{\frac{n}{90}} \quad \sqrt{\frac{90-n}{90}})$, where n is the (unknown) number of black balls, corresponds to betting on black. It is easily checked that the expected utilities are $U_{c_1} = 1/3$ and $U_{c_2} = n/90$. If we assume that n is uniformly distributed between 0 and 60 then the expected value of the latter is 1/3 as in the classical case.

Following the same procedure, we now assume that the utility matrix is of the form

$$U = \begin{pmatrix} 100 & \alpha \\ \alpha & 0 \end{pmatrix}$$

which gives $U_{c_1} = 1/3 + 0.94\alpha$ and $U_{c_2} = n/90 + 2\alpha\sqrt{\frac{n(90-n)}{90}}$ which averages to $U_{c_2} = 1/3 + 0.83\alpha$. As with the Allais case above, this means that the utility of an outcome is measured not just in terms of the expected payoff, but also reflects a subjective balance between the events of winning and losing. If $\alpha = 0$ then the two bets are equally attractive, but if $\alpha > 0$, which in this case reflects avoidance towards uncertainty, then the second option becomes more attractive.

This method of introducing non-zero off-diagonal entries to the utility matrix U has the benefit of simplicity, however a possible drawback is that U combines two effects: the objective weights, and the subjective weights. Rather than starting with U , an alternative method would be to propose a subjective basis on which to first project the state ψ , and use the spectral decomposition together with normalization conditions to generate U .¹³ This would probably require using a numerical algorithm to find the right basis.

¹² Blutner & Graben, 2014. <https://arxiv.org/ftp/arxiv/papers/1410/1410.3961.pdf>

¹³ Aerts, D., & Sozzo, S. (2016). From ambiguity aversion to a generalized expected utility. Modeling preferences in a quantum probabilistic framework. *Journal of Mathematical Psychology* 74, 117-127.

The conjunction effect

The above examples show how the projective nature of quantum probability can be used to address some of the problems with the classical approach. Other classes of cognitive effects include the disjunction effects and conjunction effects. In logic, disjunction refers to the OR statement and conjunction to AND. An example of the disjunction effect is an experiment from Tversky and Shafir, who asked a group of students the following question:¹⁴

Imagine that you have just taken a tough qualifying examination. It is the end of the fall quarter, you feel tired and run-down, and you are not sure that you passed the exam. In case you failed you have to take the exam again in a couple of months – after the Christmas holidays. You now have an opportunity to buy a very attractive 5-day Christmas vacation package in Hawaii at an exceptionally low price. The special offer expires tomorrow, while the exam grade will not be available until the following day. Would you?:

- (a) buy the vacation package. 32%
- (b) not buy the vacation package. 7%
- (c) pay a \$5 non-refundable fee in order to retain the rights to buy the vacation package at the same exceptional price the day after tomorrow – after you find out whether or not you passed the exam.

In this experiment, 32 percent choose option (a), 7 percent option (b), and a majority of 61 percent go for option (c). These results can be compared with a second version of the experiment which had exactly the same wording, except that now the students were told whether they had passed the exam or not. In this case more than half the students chose to buy the vacation package if they knew the outcome, whether it was pass (54 percent chose to buy) or fail (57 percent). However only 32 percent chose to buy the package if they didn't know the outcome.

The approach here is based on (Agrawal & Sharda, 2010), which in turn was based on the quantum decision theory of Yukalov and Sornette, discussed below.¹⁵ Let B represent the event of buying the vacation, A the event of passing the exam, and \bar{A} the event of failing the

¹⁴ Tversky & Shafir, 1992.

¹⁵ Agrawal & Sharda, 2010, Quantum Mechanics and Human Decision Making

exam. The probability of buying the vacation if the test is passed is $P(B|A) = 0.54$, and if the test is failed it is $P(B|\bar{A}) = 0.57$. However the probability of buying the vacation without knowing the test result is $P(B) = 0.32$. The law of total probability states that

$$P(B) = P(B|A)P(A) + P(B|\bar{A})P(\bar{A}).$$

In other words, the probability of B should be the same whether the result of the exam is A or \bar{A} . Applying it here, since $P(B|A) > 0.5$ and $P(B|\bar{A}) > 0.5$, we find that

$$P(B) > 0.5(P(A) + P(\bar{A})) = 0.5$$

which contradicts the fact that $P(B) = 0.32$.

Tversky & Shafir (1992) explained the disjunction effect as being caused by “the loss of acuity induced by the presence of uncertainty.” In other words, people get confused. However Busemeyer and Bruza (2012, p. 267) noted the similarity with the quantum concept of interference: “If choice is based on reasons, then the unknown condition has two good reasons. Somehow these two good reasons cancel out to produce no reason at all! This is analogous to wave interference where two waves meet with one wave rising while the other wave is falling so they cancel out.”

The disjunction effect differs from the Allais and Ellsberg paradoxes, because the question is no longer about choosing between risk/payoff balances. The payoff is not a choice between amounts of money, weighted by a probability, but between taking a vacation or staying at home. The question is therefore about the likelihood of choosing one option, given a particular context. In quantum terms, one can think of this as two projections: one which projects the context onto a particular subspace (pass or fail the test) and another which projects the vacation decision onto yes or no. The notation for this problem is summarised in Tables 1 and 2 below.

Table 1. Summary of probabilities. Experimental values are shown in bold. There are similar expressions for the event \bar{A} of failing the test and the event \bar{B} of not buying the vacation.

A	Event of passing the test
B	Event of buying vacation
$P(A)$	Probability of passing test (0.5)
$P(B A)$	Probability of buying vacation after finding test passed (0.54)
$P(BA)$	Joint probability of buying vacation and passing test $P(BA) = P(B A)P(A)$ (0.27)

$P(B)$	Probability of buying vacation without knowing test result (0.32)
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Table 2. Summary of quantum notation. There are similar expressions for $|\bar{A}\rangle$ and $|\bar{B}\rangle$.

$ A\rangle$	State where the test is passed
$ B\rangle$	State where vacation is bought
$ BA\rangle$	State where test is passed and vacation bought, $ BA\rangle = B\rangle A\rangle$

The state of the test result is the superposition

$$|\varphi\rangle = \alpha_1|A\rangle + \alpha_2|\bar{A}\rangle$$

where the coefficients are complex numbers that satisfy the normalization condition $|\alpha_1|^2 + |\alpha_2|^2 = 1$. We define \hat{O}_1 to be the operator for the probability of passing the test, with $\hat{O}_1|A\rangle = |A\rangle$ and $\hat{O}_1|\bar{A}\rangle = 0$. Then $\hat{O}_1|\varphi\rangle = \alpha_1\hat{O}_1|A\rangle + \alpha_2\hat{O}_1|\bar{A}\rangle = \alpha_1|A\rangle$. The probabilities are then $P(A) = \langle\varphi|\hat{O}_1|\varphi\rangle = |\alpha_1|^2$, and $P(\bar{A}) = 1 - |\alpha_1|^2 = |\alpha_2|^2$.

The person's state of mind after learning they passed the test is

$$|\psi_1\rangle = \gamma_{11}|BA\rangle + \gamma_{12}|\bar{B}A\rangle$$

with $|\gamma_{11}|^2 + |\gamma_{12}|^2 = 1$. We again define \hat{O}_2 to be the operator for the probability of buying the vacation, with $\hat{O}_2|B\rangle = |B\rangle$ and $\hat{O}_2|\bar{B}\rangle = 0$. The probability of buying or not buying a vacation is $P(B|A) = \langle\psi_1|\hat{O}_2|\psi_1\rangle = |\gamma_{11}|^2$, and $P(\bar{B}|A) = |\gamma_{12}|^2$.

Similarly, the person's state of mind after learning they failed the test is

$$|\psi_2\rangle = \gamma_{21}|B\bar{A}\rangle + \gamma_{22}|\bar{B}\bar{A}\rangle$$

with $|\gamma_{21}|^2 + |\gamma_{22}|^2 = 1$. The probability of buying or not buying is $P(B|\bar{A}) = \langle\psi_2|\hat{O}_2|\psi_2\rangle = |\gamma_{21}|^2$, and $P(\bar{B}|\bar{A}) = |\gamma_{22}|^2$.

Combining these, the state of mind towards the vacation before knowing the test result is

$$\begin{aligned} |\psi\rangle &= \alpha_1|\psi_1\rangle + \alpha_2|\psi_2\rangle = \alpha_1(\gamma_{11}|BA\rangle + \gamma_{12}|\bar{B}A\rangle) + \alpha_2(\gamma_{21}|B\bar{A}\rangle + \gamma_{22}|\bar{B}\bar{A}\rangle) \\ &= \beta_{11}|BA\rangle + \beta_{12}|\bar{B}A\rangle + \beta_{21}|B\bar{A}\rangle + \beta_{22}|\bar{B}\bar{A}\rangle \end{aligned}$$

where $\beta_{ij} = \alpha_i\gamma_{ij}$. Since this is a sum of orthogonal terms, $|\psi\rangle$ exists in a 4-dimensional space. The joint probability of passing the test and buying the vacation is described by the product of \hat{O}_1 and \hat{O}_2 , so

$$P(BA) = \langle \psi | \hat{O}_2 \hat{O}_1 | \psi \rangle.$$

Now

$$\hat{O}_1 |\psi\rangle = \beta_{11} \hat{O}_1 |BA\rangle + \beta_{12} \hat{O}_1 |\bar{B}A\rangle + \beta_{21} \hat{O}_1 |B\bar{A}\rangle + \beta_{22} \hat{O}_1 |\bar{B}\bar{A}\rangle = \beta_{11} |BA\rangle + \beta_{12} |\bar{B}A\rangle$$

and

$$\hat{O}_2 \hat{O}_1 |\psi\rangle = \beta_{11} \hat{O}_2 |BA\rangle + \beta_{12} \hat{O}_2 |\bar{B}A\rangle = \beta_{11} \hat{O}_2 |BA\rangle$$

so

$$P(BA) = |\beta_{11}|^2.$$

Note that $P(BA) = |\beta_{11}|^2 = |\alpha_1|^2 |\gamma_{11}|^2 = P(A)P(B|A)$ as expected. Similarly $P(\bar{B}A) = |\beta_{12}|^2$, $P(B\bar{A}) = |\beta_{21}|^2$, and $P(\bar{B}\bar{A}) = |\beta_{22}|^2$. It is easily checked that because the terms are orthogonal,

$$P(B) = \langle \psi' | \hat{O}_2 | \psi' \rangle = |\beta'_{11}|^2 + |\beta'_{21}|^2 = P(BA) + P(B\bar{A}).$$

The corresponding state for the scenario where there is no knowledge of the test result is given by the same equation where the effect of A and \bar{A} is ignored, so

$$|\psi'\rangle = \beta'_{11} |B\rangle + \beta'_{12} |\bar{B}\rangle + \beta'_{21} |B\rangle + \beta'_{22} |\bar{B}\rangle = (\beta'_{11} + \beta'_{21}) |B\rangle + (\beta'_{21} + \beta'_{22}) |\bar{B}\rangle.$$

The coefficients satisfy $|\beta'_{ij}| = |\beta_{ij}|$ but may differ in phase factor. The probability is then

$$\begin{aligned} P(B) &= \langle \psi' | \hat{O}_2 | \psi' \rangle = |\beta'_{11} + \beta'_{21}|^2 = |\beta'_{11}|^2 + |\beta'_{21}|^2 + \beta'^*_{11} \beta'_{21} + \beta'^*_{21} \beta'_{11} \\ &= P(BA) + P(B\bar{A}) + q(B) \end{aligned}$$

where $q(B) = \beta'^*_{11} \beta'_{21} + \beta'^*_{21} \beta'_{11}$ is an interference term and $*$ denotes complex conjugate.

Writing the coefficients as $\beta'_{11} = |\beta'_{11}| \exp(i\theta_{11})$ and $\beta'_{21} = |\beta'_{21}| \exp(i\theta_{21})$ gives

$$\begin{aligned} q(B) &= 2|\beta'_{11}||\beta'_{21}| \exp(i\theta_{11}) \exp(i\theta_{21}) = 2|\beta'_{11}||\beta'_{21}| \cos(\theta_{21} - \theta_{11}) \\ &= 2\sqrt{P(BA)P(B\bar{A})} \cos(\theta_{21} - \theta_{11}). \end{aligned}$$

A similar argument using the operator $\hat{I} - \hat{O}_2$, where \hat{I} is the identity operator, gives

$$P(\bar{B}) = P(\bar{B}A) + P(\bar{B}\bar{A}) + q(\bar{B})$$

where

$$q(\bar{B}) = 2\sqrt{P(\bar{B}A)P(\bar{B}\bar{A})} \cos(\theta_{22} - \theta_{12}).$$

Note that

$$\begin{aligned} q(B) + q(\bar{B}) &= (P(B) - P(BA) - P(B\bar{A})) + (P(\bar{B}) - P(\bar{B}A) - P(\bar{B}\bar{A})) \\ &= (P(B) + P(\bar{B})) - (P(BA) + P(B\bar{A}) + P(\bar{B}A) + P(\bar{B}\bar{A})) = 0. \end{aligned}$$

From the experimental results we have $P(A) = P(\bar{A}) = 0.5$, $P(B|A) = 0.54$, $P(B|\bar{A}) = 0.57$, and $P(B) = 0.32$. We therefore deduce that

$$q(B) = P(BA) + P(B\bar{A}) - P(B) = P(B|A)P(A) + P(B|\bar{A})P(\bar{A}) - P(B) = 0.235$$

and

$$\cos(\theta_{21} - \theta_{11}) = \frac{q(B)}{2\sqrt{P(BA)P(B\bar{A})}} = \frac{0.235}{2\sqrt{0.5 * 0.54 * 0.5 * 0.57}} = 0.424$$

so the phase difference is

$$\Delta = \theta_{21} - \theta_{11} = \arccos(0.424) = 64.9^\circ.$$

To summarise, quantum probability differs from classical probability in a number of respects. In a classical model, we can get the probability of an event by summing the probability of alternative paths, so $P(B) = P(B|A) + P(B|\bar{A})$. In quantum probability, probability amplitudes are first summed, and then squared to give the actual probability. Furthermore the coefficients can be complex, which leads to interference effects which do not occur in classical theory. In quantum theory, this occurs when the two events are entangled.¹⁶ This is why the quantum approach can address problems such as the disjunction effect which elude a classical treatment.

Density matrix

So far we have assumed that decisions are taken independently, so the decision maker can be modeled as being in a pure quantum state, and described using a wave function. When decision makers are interacting with society, this is no longer possible, because in effect they will be just one part of a larger societal wave function. In quantum mechanics such “mixed states” are handled by using statistical operators.

A first step is to define the density matrix or density operator, which is an alternate representation of the state of a quantum system.¹⁷ It can be calculated from a wavefunction by taking the outer product of the wavefunction and its conjugate: $\rho = |\psi\rangle\langle\psi|$.

If $|\psi\rangle = \alpha_1|\psi_1\rangle + \alpha_2|\psi_2\rangle$, then

¹⁶ Yukalov & Sornette, Conditions for Quantum Interference in Cognitive Sciences

¹⁷ https://ocw.mit.edu/courses/chemistry/5-74-introductory-quantum-mechanics-ii-spring-2009/lecture-notes/MIT5_74s09_lec12.pdf

$$\rho = |\psi\rangle\langle\psi| = \alpha_1\alpha_1^*|\psi_1\rangle\langle\psi_1| + \alpha_1\alpha_2^*|\psi_1\rangle\langle\psi_2| + \alpha_2\alpha_1^*|\psi_2\rangle\langle\psi_1| + \alpha_2\alpha_2^*|\psi_2\rangle\langle\psi_2|$$

and in general the elements of the density matrix are given by $\rho_{n,m} = \alpha_n\alpha_m^*$.

Suppose \hat{R} is an observable of the system. Then the expectation value of \hat{R} is given by

$$\begin{aligned}\langle\hat{R}\rangle &= \langle\psi|\hat{R}|\psi\rangle = \alpha_1\alpha_1^*\langle\psi_1|\hat{R}|\psi_1\rangle + \alpha_1\alpha_2^*\langle\psi_1|\hat{R}|\psi_2\rangle + \alpha_2\alpha_1^*\langle\psi_2|\hat{R}|\psi_1\rangle + \alpha_2\alpha_2^*\langle\psi_2|\hat{R}|\psi_2\rangle \\ &= \sum_{n,m} R_{n,m}\rho_{n,m} = \text{Tr}(R\rho)\end{aligned}$$

so the expectation value can be calculated by taking the trace of a matrix product. For the example of the conjunction effect, if

$$\psi = \begin{pmatrix} \sqrt{0.8} \\ 0 \\ \sqrt{0.2} \end{pmatrix}$$

then the density matrix is

$$\rho = \begin{pmatrix} \sqrt{0.8} \\ 0 \\ \sqrt{0.2} \end{pmatrix} \begin{pmatrix} \sqrt{0.8} & 0 & \sqrt{0.2} \end{pmatrix} = \begin{pmatrix} 0.8 & 0 & \sqrt{0.16} \\ 0 & 0 & 0 \\ \sqrt{0.16} & 0 & 0.2 \end{pmatrix}.$$

The density matrix representation will be used below when we account for social effects.

Projected expected utility

The disjunction problem can also be expressed in a compact way using La Mura's theory of projected expected utility. Following (Blutner & Graben, 2014) we first denote the operator associated with the test result as a projection matrix \mathbf{A} , the operator associated with the vacation decision as \mathbf{B} , and the operator which represents taking two sequential measurements \mathbf{A} then \mathbf{B} as $(\mathbf{A}; \mathbf{B})$. Consider the state $v = \mathbf{A}(u)$. The probability of $\mathbf{B}(v)$ is given by the squared amplitude

$$P_v(\mathbf{B}) = |\mathbf{B}(v)|^2 = \mathbf{B}(\mathbf{A}(u)) \cdot \mathbf{B}(\mathbf{A}(u)) = \mathbf{B}\mathbf{A}(u) \cdot \mathbf{B}\mathbf{A}(u) = \mathbf{A}(u) \cdot \mathbf{B}\mathbf{B}\mathbf{A}(u)$$

Using the fact that \mathbf{A} and \mathbf{B} are self-adjoint projection operators, we have $\mathbf{B}\mathbf{B} = \mathbf{B}$ and $\mathbf{A}(u) = u \cdot \mathbf{A}$. Plugging these into the above gives $|\mathbf{B}(v)|^2 = u \cdot \mathbf{A}\mathbf{B}\mathbf{A}(u)$ from which it follows that the sequence operator is $(\mathbf{A}; \mathbf{B}) = \mathbf{A}\mathbf{B}\mathbf{A}$.¹⁸ It can be used to calculate conditional probabilities by noting that $P(\mathbf{A}; \mathbf{B}) = P(\mathbf{B}|\mathbf{A})P(\mathbf{A})$. If \mathbf{A} and \mathbf{B} commute, then $(\mathbf{A}; \mathbf{B}) = \mathbf{A}\mathbf{B}\mathbf{A} = \mathbf{B}\mathbf{A}\mathbf{A} = \mathbf{B}\mathbf{A}$ which is the same as in the classical case.

¹⁸ Niestegge (2008). See also Busemeyer & Bruza, 2012: 365.

This operator $(\mathbf{A}; \mathbf{B})$ is known as the asymmetric conjunction operator. To see why, using the facts that $\bar{\mathbf{A}} = \mathbf{I} - \mathbf{A}$ where \mathbf{I} is the identity, and $\bar{\mathbf{A}}\mathbf{A} = \mathbf{A}\bar{\mathbf{A}} = \mathbf{0}$, then another projection operator \mathbf{B} can be expressed as

$$\begin{aligned}\mathbf{B} &= \mathbf{I}\mathbf{B}\mathbf{I} = (\mathbf{A} + \bar{\mathbf{A}})\mathbf{B}(\mathbf{A} + \bar{\mathbf{A}}) = \mathbf{A}\mathbf{B}\mathbf{A} + \mathbf{A}\mathbf{B}\bar{\mathbf{A}} + \bar{\mathbf{A}}\mathbf{B}\mathbf{A} + \bar{\mathbf{A}}\mathbf{B}\bar{\mathbf{A}} \\ &= (\mathbf{A}; \mathbf{B}) + (\bar{\mathbf{A}}; \mathbf{B}) + \partial(\mathbf{A}, \mathbf{B}).\end{aligned}$$

This is similar to the law of total probability, except that there is now an additional term $\partial(\mathbf{A}, \mathbf{B}) = \mathbf{A}\mathbf{B}\bar{\mathbf{A}} + \bar{\mathbf{A}}\mathbf{B}\mathbf{A}$ which is the interference term. Because the three terms are all mutually orthogonal, we therefore have

$$P_u(\mathbf{B}) = P_u(\mathbf{A}; \mathbf{B}) + P_u(\bar{\mathbf{A}}; \mathbf{B}) + P_u(\partial(\mathbf{A}, \mathbf{B}))$$

and applying the definition of conditional probabilities gives

$$P_u(\mathbf{B}) = P_u(\mathbf{B}|\mathbf{A})P_u(\mathbf{A}) + P_u(\mathbf{B}|\bar{\mathbf{A}})P_u(\bar{\mathbf{A}}) + P_u(\partial(\mathbf{A}, \mathbf{B})).$$

If we assume that the system is in a pure state u , then it can be shown that the probability function for the interference term is given by

$$P_u(\partial(\mathbf{A}, \mathbf{B})) = 2\sqrt{P_u(\mathbf{B}|\mathbf{A})P_u(\mathbf{A}; \mathbf{B})} \cdot \sqrt{P_u(\mathbf{B}|\bar{\mathbf{A}})P_u(\bar{\mathbf{A}})} \cdot \cos(\Delta)$$

which is the same as the result for $q(B)$ found above.¹⁹

The conjunction effect

Similar to the disjunction effect is the conjunction effect. This was illustrated in a 1983 paper by Tversky and Kahneman, who described an experiment where they gave participants the following profile: “Linda is 31 years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations.” They then asked for the odds that (A) Linda is active in the feminist movement, (B) Linda is a bank teller, and finally that (A&B) Linda is a bank teller AND is active in the feminist movement. The results were the probability $P(A) = 0.61$ for feminist, $P(B) = 0.38$ for bank teller, and $P(A; B) = 0.51$ for both.

According to the rule of conjunction in probability theory, the probability of two events occurring together must be less than or equal to the probability of either one occurring alone. However here the probability of A and B occurring together is greater than the probability of B alone. It is inconsistent with classical logic, but again appears in the quantum model as an

¹⁹ Blutner & Graben, 2014.

interference term. The prospect “feminist bank teller” can be written in projected utility theory as $(A; B)$, while the joint prospect “bank teller” is $B = B(A + \bar{A})$. The conjunction effect is defined as the term $P(A; B) - P(B) = 0.13$. A similar analysis to the above gives $\cos(\Delta) = -0.9$, so $\Delta = 154^\circ$.²⁰

The order effect

A very simple example of the role of interference in quantum probability is supplied by the so-called order effect. As discussed in the book, pollsters and survey writers have long known that the answers they receive depend on the exact wording of the questions, but also on their order. The response to the first question changes the context for the second question, where here the context includes the responder’s own state of mind. The phenomenon is so common that in psychology “non-commutativity should be the ubiquitous rule,” according to psychologists Harald Atmanspacher and Hartmann Römer. It makes no sense from the point of view of classical utility theory, but is similar to the one encountered in quantum physics, where a measurement of position affects a particle’s momentum and vice versa.

In a 2014 paper, researchers analysed the results of 70 US surveys, and found that the way the answers changed showed an underlying symmetry.²¹ One example they used was a Gallup survey from 1997 which asked in consecutive questions whether respondents thought Bill Clinton and Al Gore were trustworthy. The number of people who described them both as trustworthy was 49 per cent if Clinton was named first, but rose to 56 per cent if Gore was named first, a difference of 7 per cent. Conversely, the number who described them both as untrustworthy was 28 per cent if Clinton was named first, but fell to 21 per cent if Gore was named first, again a difference of 7 per cent. So the increase in joint trustworthiness was balanced by a decrease in joint untrustworthiness.

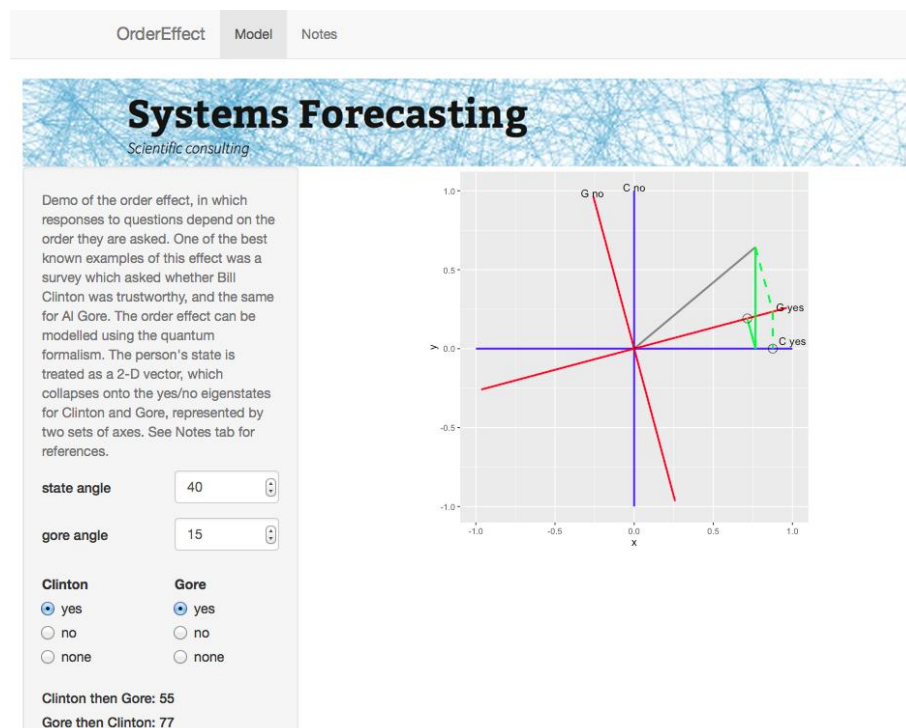
We can express the order effect through the interference term

$$\partial(A, B) = B - (A; B) - (\bar{A}; B)$$

²⁰ Blutner & Graben, 2014.

²¹ Wang, Z., Solloway, T., Shiffrin, R.S., and Busemeyer, J.R. (2014), ‘Context effects produced by question orders reveal quantum nature of human judgments’, *Proceedings of the National Academy of Sciences*, 111 (26), pp. 9431–6.

which in this case reflects the sensitivity of the response to a question **B** where the context is set by a response to another question **A**.²² However the quantum model for this experiment is very simple and can be visualised without the use of equations.²³ The screenshot below is from a web application which is available online.²⁴ The grey line shows the person's state when answering the Clinton question. It therefore represents a snapshot of a probabilistic wave function, which is in a superposition of two states, trust and mistrust. If the person was sure of their trust in Clinton, then this line would align closely with the horizontal YES axis; if they were very distrustful, it would align with the vertical NO axis. This person is rather unsure so holds the two options in superposition with roughly equal strength, and the line is nearly diagonal. A decision to answer yes is equivalent to a collapse of the uncertain superposed state, and is represented mathematically by projecting onto the YES axis, to the point shown by the white circle. The probability of this choice, according to the quantum model, is then the square of the distance of that point from the centre. This collapsed state is then used as the initial condition for the answer to the next question. It is seen that changing the question order affects the response probability, in a way that respects the symmetry between joint trustworthiness and joint untrustworthiness.



²² Blutner & Graben, 2014.

²³ For the 2-D case the coefficients can be assumed to be real rather than complex, see Moreira, C. & Wichert, A. (2017), 'Are Quantum Models for Order Effects Quantum?', *International Journal of Theoretical Physics* 56(12): 4029–4046.

²⁴ See: <https://david-systemsforecasting.shinyapps.io/ordereffect/>

Figure 3.3. Screenshot of the order effect demo, available as a web application.

Preference reversal

As seen with the examples above, the classical and quantum approaches to probability are very different. The former involves sets, while the latter involves subspaces that are defined by projection operators which need not commute. Classical probabilities are additive, but quantum probabilities are computed by taking squared amplitudes. These effects become particularly important when considering problems that involve decisions under uncertainty, of the same kind considered in prospect theory. An example is the phenomenon of preference reversal.

Preference reversal is when we prefer a particular prospect in one context, but not under a different context. Since the choice of prospects is unchanged, this suggests that the subjective basis is context-dependent. It is therefore different from the disjunction effect, where the comparison is between projecting options onto a subjective basis after being given information (for the vacation example, about a test result) or evaluating them directly.

To model preference reversal, we will focus on the quantum decision theory (QDT) approach of Yukalov and Sornette, which is convenient for analysing more complex prospects. The theory has been explained in a series of publications, and applied to various cases including those considered above, so we will just offer a brief summary here.²⁵

In QDT problems are written in terms of prospects of the form $\pi_1 = BA$, which corresponds to $|\pi_1\rangle = a_{11}|BA\rangle$ in Dirac notation.²⁶ The operators which measure the system (e.g. \hat{O}_1 and \hat{O}_2 in the conjunction effect example test whether the exam was passed or the vacation bought) are replaced by “prospect operators” of the form $\hat{R}(\pi_1) = |a_{11}|^2|BA\rangle\langle BA|$. However the results are exactly the same.²⁷ For example if $|\pi_2\rangle = a_{21}|BA\rangle + a_{22}|B\bar{A}\rangle$, Yukalov & Sornette show that

²⁵ Yukalov & Sornette, Mathematical Structure of Quantum Decision Theory, https://papers.ssrn.com/sol3/papers.cfm?abstract_id=1263853

²⁶ Kovalenko & Sornette, The conjunction fallacy in quantum decision theory, p. 6.

²⁷ See Agrawal & Sharda, Quantum Mechanics and Human Decision Making.

$$P(\pi_2) = P(BA) + P(B\bar{A}) + Q(BA) = P(B|A)P(A) + P(B|\bar{A})P(B\bar{A}) + q(BA)$$

where $q(BA)$ is the interference term, which is equivalent to the one obtained above using other methods. In the density matrix representation, the prospect probability is equal to the normalised expectation value of the prospect operator $\hat{R}(\pi_n)$, so $P(\pi_n) = \text{Tr} \hat{\rho} \hat{R}(\pi_n)$.

They write this as

$$P(\pi_2) = f(\pi_2) + g(\pi_2)$$

where the first term is called a utility factor, and the second term an attraction factor which represents subjective biases. The former corresponds to the diagonal elements of the prospect, and the latter to the off-diagonal elements. As they note, “It is the appearance of interference terms that makes the structure of quantum expressions richer than the related classical ones and that allows one to explain those psychological phenomena that, otherwise, are inexplicable in classical decision making.”²⁸

As a simple example, consider the mental state of a person who is faced with two alternative prospects, denoted A_1 and A_2 . The person’s attitude towards these prospects will be shaped by subjective desires which we call B_1 and B_2 (there can be any number). The two joint prospects can be expressed in a Hilbert space as the superposition states

$$\begin{aligned} |\pi_1\rangle &= \gamma_{11}|A_1B_1\rangle + \gamma_{12}|A_1B_2\rangle \\ |\pi_2\rangle &= \gamma_{21}|A_2B_1\rangle + \gamma_{22}|A_2B_2\rangle \end{aligned}$$

where the coefficients γ_{ij} can again be complex. In the notation of projected utility, this is $\pi_1 = (B_1; A_1) + (B_2; A_1)$ and $\pi_2 = (B_1; A_2) + (B_2; A_2)$.

Since the context affects the subjective basis, we need to model the person’s state in the general form prior to being presented with the problem. This is the superposition

$$|\psi\rangle = \alpha_{11}|A_1B_1\rangle + \alpha_{12}|A_1B_2\rangle + \alpha_{21}|A_2B_1\rangle + \alpha_{22}|A_2B_2\rangle$$

where the coefficients satisfy

$$|\langle\psi|\psi\rangle|^2 = |\alpha_{11}|^2 + |\alpha_{12}|^2 + |\alpha_{21}|^2 + |\alpha_{22}|^2 = 1.$$

The probability of the person choosing the prospect π_j is therefore

$$p(\pi_j) = \frac{1}{P} |\langle\psi|\pi_j\rangle|^2 = \frac{1}{P} (\alpha_{j1}^* \gamma_{j1} + \alpha_{j2}^* \gamma_{j2}) (\alpha_{j1} \gamma_{j1}^* + \alpha_{j2} \gamma_{j2}^*)$$

where $P = |\langle\psi|\pi_1\rangle|^2 + |\langle\psi|\pi_2\rangle|^2$ is a normalisation term to ensure that the probabilities add to 1.

²⁸ Yukalov & Sornette, Preference reversal in quantum decision theory

This can be written in the form

$$p(\pi_j) = f(\pi_j) + q(\pi_j)$$

where

$$f(\pi_j) = \frac{1}{P} (|\alpha_{j1}|^2 |\gamma_{j1}|^2 + |\alpha_{j2}|^2 |\gamma_{j2}|^2)$$

is called the utility function, and

$$q(\pi_j) = \frac{1}{P} (\alpha_{j1}^* \gamma_{j1} \alpha_{j2} \gamma_{j2}^* + \alpha_{j2}^* \gamma_{j2} \alpha_{j1} \gamma_{j1}^*)$$

is called the attraction function. The utility function separates out the two terms corresponding to the outcomes A_1 and A_2 (in the lottery example it would correspond to the expected payout from the lottery), while the attraction function represents their entanglement through the different subjective contexts B_1 and B_2 . Note that if there is no entanglement, then $q(\pi_j) = 0$, the probabilities are the same as for the classical approach, and there is no need to evoke quantum methods.

Since a classical utility term is in the form of a probability term, we need to have $f(\pi_1) + f(\pi_2) = 1$. But since $p(\pi_1) + p(\pi_2) = 1$ it follows that $q(\pi_1) + q(\pi_2) = 0$. It can then be shown that, in the absence of any information about the structure of the attraction function, we can expect the attraction function of the more attractive choice to be $\frac{1}{4}$, and the less attractive choice to be $-\frac{1}{4}$. This result is known as the “quarter law” and has been tested empirically in a variety of situations using controlled experiments.²⁹

Now, according to classical utility theory, the person is expected to choose prospect π_1 if $f(\pi_1) - f(\pi_2) > 0$. In QDT however we have to take into account the interference terms, so the relevant test becomes $f(\pi_1) + q(\pi_1) - f(\pi_2) - q(\pi_2) > 0$, or equivalently $f(\pi_1) - f(\pi_2) > 2|q(\pi_1)|$. In other words, the attraction function sets a threshold which needs to be exceeded in order for an option to be seen as preferable. Following the quarter law, a starting guess is that the utility (on a scale of 0 to 1) of an option has to exceed that of the other one by 0.5. Yukalov and Sornette call this the preference reversal criterion, for reasons discussed below.

²⁹ Yukalov & Sornette, 2015.

Put another way, suppose that the more attractive option has an associated cost x_1 and the less attractive option has a cost x_2 . We can assign the relative utility functions

$$f(\pi_1) = \frac{x_2}{x_1 + x_2}$$

$$f(\pi_2) = \frac{x_1}{x_1 + x_2}$$

which sum to 1. The preference reversal condition is then

$$f(\pi_2) - f(\pi_1) = \frac{x_2 - x_1}{x_1 + x_2} > \frac{1}{2}.$$

Equality in the above expression is attained if $x_2 = 3x_1$, and in general if the condition holds we might expect $\frac{x_2}{x_1} > 3$. Again, this should only be viewed as a first approximation, but highlights the significant role that subjective effects play in decision making.

Preference reversal with mortgage default

Quantum decision theory has so far mostly been applied to experiments where participants are asked to choose between carefully crafted lotteries with different balances of risk and reward. As mentioned above, one example is preference reversal, where the choice is typically between two lotteries, the first offering a high probability of a low payout, the second a low probability of a high payout. If the expected utility of the second lottery is a little higher, then people still tend to choose the first lottery for themselves. But if the question is reframed so they are asked to price a ticket which can be sold to someone else, they value the second lottery more highly.

Tversky and Thaler (1990) determined that the phenomenon is caused by the breaking of procedure invariance: subjects weight payoffs in pricing more heavily than in choice. They conclude: “First, people do not possess a set of pre-defined preferences for every contingency. Rather, preferences are constructed in the process of making a choice or judgment. Second, the context and procedures involved in making choices or judgments influence the preferences that are implied by the elicited responses. In practical terms, this implies that behavior is likely to vary across situations that economists consider identical.”

It seems that we use one mental framework when making a choice, and another when assessing a price, in a manner that can be addressed in a classical model only by introducing

ad hoc weighting factors. The quantum approach however has this context sensitivity built in by default. A person's mental state is modelled, not in terms of fixed preferences, but as a context-dependent wave function which collapses to a decision only when a question is asked.

Yukalov and Sornette (2015) analysed experimental data sets for such lottery examples to show that the transition point between the two choices follows their preference reversal criterion. They also relate it to the so-called planning paradox, where we prefer one thing when we are talking about the future, and another when we are talking about the present. An example is the situation of a smoker who is deciding on whether to stop smoking now, or stop smoking later. In terms of expected utility they should be the same, but in practice the latter is much more attractive, which is why people find it hard to quit.

A more economically relevant application is the case of default among mortgage holders.³⁰ Usually this occurs because factors such as unemployment or divorce mean that the homeowner can no longer afford the mortgage payments. However, if house prices have declined so that the home is worth less than the mortgage, then the homeowner may also decide to walk away, which is known as strategic default.

Guiso et al. (2009) used data from a quarterly survey of a representative sample of U.S. households from December 2008 to September 2010 in order to determine the attitude of homeowners toward strategic default. The results showed that roughly 30% of respondents said they would default if the shortfall was more than \$100K, and a 64% majority said they would default if it exceeded \$200K. However the actual statistics for foreclosure paint a very different picture. By mid-2009 over 16% of U.S. homeowners had negative equity exceeding 20% of their home's value, and over 22% of homeowners had negative equity exceeding 10% of their home's value.³¹ Given the high value of homes in the most-affected markets, many of these homeowners were underwater by well over \$100K. If 30% of these had opted for strategic default, in accordance with the survey results, it would have represented in total around 5% of American homeowners. However, while by the third quarter of 2009 the combined foreclosure and thirty-plus-day delinquency rate for home mortgages did reach a

³⁰ See the discussion paper "Quantum Financial Entanglement: The Case of Strategic Default" at https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3394550, and references therein.

³¹ White, 2010.

historic high of 14%, only a small fraction of these were strategic. Bradley et al. (2015) estimated the proportion to be in the range of 7.7% to 14.6%, which would put the overall strategic default rate at only 1% to 2%. According to one estimate from the Federal Reserve “the median borrower does not strategically default until equity falls to -62 percent of their home’s value.”³²

On the face of it, this behaviour seems irrational, since even given the various costs of foreclosure the best option from a narrow utilitarian point of view would often be default. Behavioural economists typically explain such effects by appealing to the idea that homeowners suffer from cognitive biases which lead them to make poor economic decisions, and behavioural models exist that fit the data by adjusting for things like present bias and discount rates. However this does not explain the fact that even when homeowners can see it makes economic sense to default – and say they would default in a survey – they usually decline to do so in practice. Instead it seems that the primary motivation for staying in the home is the desire to avoid shame and social stigma, and fear of the perceived (and often exaggerated) consequences of default (White, 2010). In other words, the response is driven not so much by cognitive deliberations but a powerful mix of emotions. And the fact that this combination of guilt and fear is felt far more keenly when actually making a decision to stay or move, as opposed to answering a survey question, is why observed default rates are far lower than one might expect from calculations based on either survey results or utility maximization.

The situation is therefore similar to the case of preference reversal described above, where we evaluate an option differently depending on whether we are making an actual choice, or coming up with a hypothetical price. The homeowner tends to prefer the perceived security of staying in their own home, even if they know it is financially suboptimal. They are also affected by their sense of morality, and social norms about the importance of honouring your debts. (The bank of course takes a very different stance, since it operates according to market norms.) The result is that, just as most smokers who say they want to quit fail to do so, so most homeowners who tell a survey they would default don’t do so in practice.

³² Bhutta, Dokko & Shan, 2010.

Instead of adjusting behavioural models, a simpler and more elegant explanation is to apply the methods of quantum decision theory, where the attraction function accounts for the context-dependent subjective factors including shame, guilt, and fear. This also has the advantage of being a consistent model which can be applied for a broad range of phenomena, as opposed to an *ad hoc* model that is tuned to fit a particular data set.

Suppose that the cost of staying in the home for a certain period is x_1 and the cost of defaulting, including renting for the same period, is x_2 . According to the preference reversal criterion, in order for strategic default to be selected we would expect the cost ratio $\frac{x_2}{x_1}$ to be around 3. The Federal Reserve estimate for the critical threshold to initiate strategic default was a 62 percent fall in value, which corresponds to a fall in utility compared to the purchase price by a factor 2.63. Given that most people would presumably choose to have their house back rather than rent, this is consistent with the quantum estimate.

Quantum decision theory, and in particular interference between objective calculations and subjective emotions, therefore helps to explain why so few people in similar situations actually chose to default, even if their behaviour seems to defy both classical utility theory, and the results of surveys: when it came to the crunch, entangled emotions such as guilt and fear interfered with and outweighed abstract considerations of utility. Perhaps the main message is that for the complex issue of strategic default, the discrepancy between utility-maximizing and observed behaviour is so large that standard calculations of utility – despite the foundational role they play in mainstream economics – are of rather little relevance.

5. The quantum harmonic oscillator

As seen above, the key idea in the quantum approach is that point objects are replaced with quantum state or wave functions, and observables are replaced with the eigenvalues of operators. This “quantization” procedure is relatively straightforward for static models like the ones considered above, but becomes considerably more complicated for dynamic systems.

One clue on how to go about this quantization procedure is the fact (first discovered by the mathematician Oliver Heaviside in the late nineteenth century) that differential operators act in some respects like ordinary numbers. Consider for example the equation

$$y + \frac{dy}{dx} = x^2.$$

Define D to be the differential operator $D = \frac{d}{dx}$, so $Dy = \frac{dy}{dx}$. Powers of D are interpreted as higher derivatives, so

$$D^2 = \frac{d^2}{dx^2}$$

and so on. Then the above equation can be written

$$(1 + D)y = x^2$$

so

$$y = \frac{x^2}{1 + D}.$$

Rewriting $\frac{1}{1+D}$ as the infinite expansion

$$\frac{1}{1 + D} = 1 - D + D^2 - D^3 \dots$$

gives

$$y = (1 - D + D^2 - D^3 \dots)x^2 = x^2 - 2x + 2$$

after applying the derivative operators to x and noting that all derivatives higher than the second are zero.

Because operators act on the object to the right of them, the two don't usually commute.

Suppose we have a function $\psi(x)$ and evaluate

$$D(x\psi) = D(x)\psi + xD(\psi) = \psi + xD(\psi)$$

so

$$D(x\psi) - xD(\psi) = D(x)\psi + xD(\psi) - xD(\psi) = \psi$$

or in operator form

$$Dx - xD = 1$$

where 1 is the identity operator that does nothing. The commutator for two elements f and g is defined as $[f, g] = fg - gf$, so here we can write $[D, x] = 1$. Such commutator relationships play an important role in quantum mechanics. One has to be careful about the order of operations, and in quantizing a system it may not be clear at first which is the correct order to use.

Now, we want to represent quantum states using wave functions. Many experiments suggest waves that have a periodicity which scales with momentum, in a manner which depends on the reduced Planck's constant \hbar . Focussing on the spatial variation, a typical wave function might therefore be of the form

$$\psi(x) = e^{-\frac{ipx}{\hbar}}.$$

In classical mechanics x would refer to a spatial coordinate, and p to a momentum. If we identify \hat{p} as the differential operator

$$\hat{p} = -i\hbar \frac{\partial}{\partial x}$$

and apply it to ψ we get

$$\hat{p}\psi = -i\hbar \frac{\partial \psi}{\partial x} = \hat{p}e^{-\frac{ipx}{\hbar}} = p\psi$$

so the observable p is an eigenvalue of the operator. We can therefore identify \hat{p} as the momentum operator. The position operator \hat{x} returns the value of x . In “momentum space” it can be defined as

$$\hat{x} = i\hbar \frac{\partial}{\partial p}$$

which has the eigenvalue x . A similar relationship (related to the requirements of relativity) holds for total energy \hat{H} and time t :

$$\hat{H} = -i\hbar \frac{\partial}{\partial t}.$$

Using the definition of the momentum operator, and the product rule of calculus, we have

$$\begin{aligned} \hat{x}\hat{p}\psi - \hat{p}\hat{x}\psi &= \hat{x}\left(-i\hbar \frac{\partial \psi}{\partial x}\right) + i\hbar \frac{\partial(\hat{x}\psi)}{\partial x} \\ &= -\hat{x}\left(i\hbar \frac{\partial \psi}{\partial x}\right) + i\hbar \left(\hat{x} \frac{\partial \psi}{\partial x} + \frac{\partial \hat{x}}{\partial x} \psi\right) = i\hbar \frac{\partial \hat{x}}{\partial x} \psi. \end{aligned}$$

But since $\frac{\partial \hat{x}}{\partial x} = 1$, it follows that $\hat{x}\hat{p}\psi - \hat{p}\hat{x}\psi = i\hbar\psi$, and the commutator therefore satisfies the relationship $[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar$. This is known as the canonical commutator relationship, which holds also for other pairs such as energy and time.

To get a better sense of how the quantization procedure works, we can apply the method to a simple physical example, which is the harmonic oscillator.³³ We choose it because it plays a key role in quantum field theory, which underpins the methods used later to describe the quantum economy; it serves as a first-order approximation to many more complicated systems; and it is one of the few quantum systems that can be solved in closed form equations.

A classical harmonic oscillator involves an object of mass m oscillating around a central point with a spring-like restoring force given by $F = -kx$, where k is a constant and x is the displacement. The equation of motion can be written in terms of momentum p as

$$\begin{aligned} p &= m\dot{x} \\ \dot{p} &= F = -kx \end{aligned}$$

or equivalently as $m\ddot{x} = -kx$. This has the oscillatory solution

$$x = A \cos(\omega t + \varphi)$$

where the phase φ depends on the starting point. The energy is given by

$$E = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

where $\omega = \sqrt{k/m}$ is the frequency of oscillation. The first term represents the kinetic energy, and the second term the potential energy.

To quantize the system, we again need to replace the classical equations with quantum versions that act on wave functions but recover the required properties of observables.³⁴ In quantum mechanics, the total energy is given by an equation known as the Hamiltonian, expressed now in terms of operators. We therefore try:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2 \hat{x}^2.$$

This can be written more simply in the form

$$\hat{H} = \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right) = \hbar\omega \left(\hat{N} + \frac{1}{2} \right)$$

where

³³ If the quantization procedure seems a little ad hoc and awkward, one reason is that we are trying to adapt classical mathematical tools to handle wave/particle duality. Another is that the approach was based on intuition and the equations were adopted, not because they can be proved to be true, but because they fit the data (which gives some latitude for social scientists to adapt them for other uses).

³⁴ I am drawing on: Barton Zwiebach. 8.05 Quantum Physics II. Fall 2013. Massachusetts Institute of Technology: MIT OpenCourseWare, <https://ocw.mit.edu>. License: Creative Commons BY-NC-SA.

$$\begin{aligned}\hat{a} &= \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} + \frac{i\hat{p}}{m\omega} \right), \\ \hat{a}^\dagger &= \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} - \frac{i\hat{p}}{m\omega} \right), \\ \hat{N} &= \hat{a}^\dagger \hat{a}.\end{aligned}$$

For reasons that will become clear, \hat{a}^\dagger is known as the creation operator, \hat{a} is the annihilation operator, and \hat{N} is the number operator. As seen by multiplying them out and using the commutator relationship between \hat{x} and \hat{p} , the creation and the annihilation operators satisfy the canonical commutator relationship with this scaling, which is

$$[\hat{a}, \hat{a}^\dagger] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1.$$

If ψ is a wave function with norm 1, then

$$\langle \psi | \hat{H} | \psi \rangle = \hbar\omega \left\langle \psi \left| \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right) \right| \psi \right\rangle = \hbar\omega \langle \hat{a}\psi | \hat{a}\psi \rangle + \frac{\hbar\omega}{2} \geq \frac{\hbar\omega}{2}$$

since any norm cannot be less than zero.

Now, suppose that $|E\rangle$ is a normalised energy state of the system. Since observables correspond to eigenvalues, it follows that $|E\rangle$ must be an eigenvector of the Hamiltonian operator, with associated eigenvalue E :

$$\hat{H}|E\rangle = E|E\rangle.$$

From this and the above inequality, we have

$$\langle E | \hat{H} | E \rangle = E \langle E | E \rangle = E \geq \frac{\hbar\omega}{2}.$$

The system therefore has a minimum energy level given by $\frac{\hbar\omega}{2}$.

Consider the two states defined as

$$|E_+\rangle = \hat{a}^\dagger |E\rangle,$$

$$|E_-\rangle = \hat{a} |E\rangle.$$

We first note that

$$[\hat{H}, \hat{a}^\dagger] = \hat{H}\hat{a}^\dagger - \hat{a}^\dagger\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a})\hat{a}^\dagger - \hat{a}^\dagger\hbar\omega(\hat{a}^\dagger\hat{a}) = \hbar\omega(\hat{a}^\dagger\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}^\dagger\hat{a})$$

since the contribution of the constant term in the Hamiltonian cancels out. Using the commutator relationship for creation and annihilation operators then gives

$$[\hat{H}, \hat{a}^\dagger] = \hbar\omega\hat{a}^\dagger[\hat{a}, \hat{a}^\dagger] = \hbar\omega\hat{a}^\dagger.$$

Similarly

$$[\hat{H}, \hat{a}] = \hbar\omega\hat{a}$$

and also

$$\hat{N}|E\rangle = \left(\frac{\hat{H}}{\hbar\omega} - \frac{1}{2}\right)|E\rangle = \hat{N}_E|E\rangle$$

where $\hat{N}_E = \frac{\hat{H}}{\hbar\omega} - \frac{1}{2}$ is the number operator eigenvalue associated with this energy state.

Then

$$\hat{H}|E_+\rangle = \hat{H}\hat{a}^+|E\rangle = ([\hat{H}, \hat{a}^+] + \hat{a}^+H)|E\rangle = (\hbar\omega + E)\hat{a}^+|E\rangle = (E + \hbar\omega)|E_+\rangle,$$

$$\hat{H}|E_-\rangle = \hat{H}\hat{a}|E\rangle = ([\hat{H}, \hat{a}] + \hat{a}H)|E\rangle = (-\hbar\omega + E)\hat{a}|E\rangle = (E - \hbar\omega)|E_-\rangle$$

so the energy state with $E_+ = E + \hbar\omega$ and $N_{E_+} = N_E + 1$ has an increased energy level, while the energy state with $E_- = E - \hbar\omega$ and $N_{E_-} = N_E - 1$ has a decreased energy level.

The reason \hat{a}^+ is called the creation operator, and \hat{a} the annihilation operator, is that these operators raise or lower the energy by $\hbar\omega$ and the number operator by one. The creation operator can always be applied to raise the energy, but the annihilation operator can only be applied to energy levels above the base level, since energy cannot be negative.

The lowest base level can be found by assuming there is a non-trivial state $|E\rangle$ that is annihilated by \hat{a} , so $\hat{a}|E\rangle = 0$. Thus $\hat{a}^+\hat{a}|E\rangle = N|E\rangle = 0$, which implies that this is the energy state with $E = \frac{\hbar\omega}{2}$ and $N_E = 0$. We can derive the equation for this state by acting with position x :

$$\langle x|\hat{a}|E\rangle = \sqrt{\frac{m\omega}{2\hbar}}\left\langle x\left|\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right)\right|E\right\rangle = 0.$$

If we define the wave function $\psi_E(x) = \langle x|E\rangle$ and use the definition of \hat{p} as a differential operator, then this gives

$$\left(x + \frac{\hbar}{m\omega}\frac{d}{dx}\right)\psi_E(x) = 0$$

or

$$\frac{d\psi_E}{dx} = -\frac{m\omega}{\hbar}x\psi_E$$

with solution

$$\psi_E(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x^2\right)$$

which is a Gaussian distribution centered at 0. The existence of this ground state reflects the uncertainty principle, in the sense that an oscillator with no energy can't exist (because then

we would know the energy is zero), and has no classical analogue. Higher energy states are more complicated, and can be determined by successively applying the creation operator.

Another way to express this is by using the number operator. Denote states as $|n\rangle$ with associated eigenvalue n , so $\hat{N}|n\rangle = n|n\rangle$. The ground state is $|0\rangle$ (which is not the same as the zero vector). The next state $|1\rangle$ is obtained by using the creation operator on $|0\rangle$,

$$|1\rangle = \hat{a}^\dagger|0\rangle$$

and

$$\hat{N}|1\rangle = \hat{a}^\dagger \hat{a} \hat{a}^\dagger|0\rangle = (\hat{a}^\dagger[\hat{a}, \hat{a}^\dagger] + \hat{a}^\dagger \hat{a}^\dagger \hat{a})|0\rangle = \hat{a}^\dagger|0\rangle = 1$$

where we have used $[\hat{a}, \hat{a}^\dagger] = 1$ and $\hat{a}|0\rangle = 0$. The equations for higher energy states can be derived recursively to give

$$|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n|0\rangle.$$

The states $|n\rangle$ form an orthonormal basis, so any state can be described in terms of a linear combination of these states.

One can also calculate the expected values of quantities for different energy levels. Some algebra using the creation and annihilation operators shows that

$$\langle n|\hat{x}|n\rangle = \sqrt{\frac{\hbar}{2m\omega}} \langle n|\hat{a} + \hat{a}^\dagger|n\rangle = 0$$

$$\langle n|\hat{p}|n\rangle = i \sqrt{\frac{m\omega\hbar}{2}} \langle n|\hat{a}^\dagger - \hat{a}|n\rangle = 0$$

$$\langle n|\hat{x}^2|n\rangle = \frac{\hbar}{2m\omega} \langle n|(\hat{a} + \hat{a}^\dagger)^2|n\rangle = \frac{\hbar}{m\omega} \left(n + \frac{1}{2}\right)$$

$$\langle n|\hat{p}^2|n\rangle = -\frac{m\omega\hbar}{2} \langle n|(\hat{a}^\dagger - \hat{a})^2|n\rangle = m\omega\hbar \left(n + \frac{1}{2}\right).$$

The uncertainties in position and momentum therefore satisfy

$$\Delta x \Delta p = \sqrt{\langle n|\hat{x}^2|n\rangle} \sqrt{\langle n|\hat{p}^2|n\rangle} = \frac{\hbar}{m\omega} \left(n + \frac{1}{2}\right) \geq \frac{\hbar}{2m\omega}$$

which is Heisenberg's uncertainty principle.

Another operator which will prove useful is the translation operator defined as

$$T_{x_0} = e^{-\frac{i}{\hbar} \hat{p} x_0}$$

which acts on a state $|\psi\rangle$ by moving it by an amount x_0 . To see this, the expectation value of \hat{x} in the state $|\psi\rangle$ is

$$\langle \hat{x} \rangle_\psi = \langle \psi | \hat{x} | \psi \rangle$$

and the expectation of \hat{x} in the state $T_{x_0}|\psi\rangle$ is

$$\langle \hat{x} \rangle_{T_{x_0}|\psi} = \langle \psi | T_{x_0}^\dagger \hat{x} T_{x_0} | \psi \rangle = \left\langle \psi \left| e^{-\frac{i}{\hbar} \hat{p} x_0} \hat{x} e^{\frac{i}{\hbar} \hat{p} x_0} \right| \psi \right\rangle.$$

The expression involving brackets can be solved to give

$$\langle \hat{x} \rangle_{T_{x_0}|\psi} = \left\langle \psi \left| \hat{x} + \frac{i}{\hbar} [\hat{p}, \hat{x}] x_0 \right| \psi \right\rangle = \hat{x} + x_0$$

as expected.³⁵

If the translation operator is applied to the ground state $|0\rangle$, then the new state is called a coherent state, and can be expressed in terms of creation and annihilation operators as follows:

$$|\hat{x}_0\rangle = T_{x_0}|0\rangle = \exp\left(-\frac{i}{\hbar} \hat{p} x_0\right) |0\rangle = \exp\left(\frac{x_0}{\sqrt{2}d} (\hat{a}^\dagger - \hat{a})\right) |0\rangle,$$

or alternatively

$$|\alpha\rangle = D(\alpha)|0\rangle$$

where $d = \sqrt{\frac{\hbar}{m\omega}}$ is a length scale, $\alpha = \frac{x_0}{\sqrt{2}d}$, and

$$D(\alpha) = \exp(\alpha \hat{a}^\dagger - \alpha^* \hat{a}) |0\rangle$$

is known as the displacement operator. When α is real, as here, the displacement is in position only, while imaginary values correspond to displacement in momentum.

Calculation shows that the total energy of the translated system is increased relative to that of the ground state by an amount $\frac{1}{2}m\omega^2 x_0^2$ which makes sense since it corresponds to the classical potential energy of a particle on a spring stretched an amount x_0 . However the system is not in a single energy state, but is of the form

$$|\hat{x}_0\rangle = \sum_{n=0}^{\infty} c_n |n\rangle.$$

³⁵ Using the Baker-Hausdorff identity $e^{\hat{A}} \hat{B} e^{-\hat{A}} = \hat{B} + [\hat{A}, \hat{B}] + \frac{1}{2!} [\hat{A}, [\hat{A}, \hat{B}]] + \dots$ where here all but the first two terms vanish.

The probability of obtaining an energy equal to E_n is $c_n^2 = \frac{\lambda^n}{n!} e^{-\lambda}$ which is a Poisson distribution with mean $\lambda = \frac{m\omega x_0^2}{2\hbar}$.

So far we have only viewed the system in a static sense. To study how the wave function $|\psi\rangle$ evolves with time, we write

$$|\psi\rangle_t = \hat{U}(t, t_0)|\psi\rangle_{t_0}$$

where $\hat{U}(t, t_0)$ is a unitary linear operator, that can be viewed as rotating the hyperspace of all possible states in the Hilbert space. Taking the derivative with respect to time gives

$$\frac{\partial}{\partial t}|\psi\rangle_t = \frac{\partial \hat{U}(t, t_0)}{\partial t}|\psi\rangle_{t_0}.$$

Using the fact (easily checked) that

$$\hat{U}(t_0, t) = \hat{U}^{-1}(t, t_0) = \hat{U}^\dagger(t, t_0)$$

then gives

$$\frac{\partial}{\partial t}|\psi\rangle_t = \frac{\partial \hat{U}(t, t_0)}{\partial t} \hat{U}^\dagger(t, t_0)|\psi\rangle_t.$$

Recalling that

$$\hat{H} = -i\hbar \frac{\partial}{\partial t}$$

gives the Schrödinger equation

$$i\frac{\partial}{\partial t}|\psi\rangle_t = \hat{H}(t)|\psi\rangle_t$$

which can be solved in a similar manner as the classical version to show that operators satisfy the same oscillatory equations of motion.

To summarise, the quantum model predicts that the observed energy levels of a harmonic oscillator are equally spaced with an interval of $\hbar\omega$ and a minimum value of $\frac{\hbar\omega}{2}$. Prior to measurement, the system will be in a superposed state of the form $|\psi\rangle = \sum_n A_n |n\rangle$, where the A_n are complex numbers, and $w_n = |A_n|^2$ is the probability that the oscillator is in the state $|n\rangle$. The evolution of the state can be solved using the Schrödinger equation.

As a physical example of the harmonic oscillator, a diatomic molecule such as the hydrogen molecule H_2 can be viewed as two atoms connected by a spring. Experimental observations show that such molecules absorb and emit photons whose frequencies are multiples of the

oscillator frequency, as expected. Many other physical systems, such as the vibration of molecules in a solid, can be similarly approximated as a system of quantum harmonic oscillators, since the assumption of a linear force can be viewed as a first-order estimate to the dynamics near the minima of a potential well. Most importantly, it turns out that the equations describing electromagnetic fields in quantum physics are like those of the harmonic oscillator, with the particles corresponding to photons, and the ground state corresponding to the energy of empty space. It is this energy that fuels the appearance of “virtual photons” which communicate the electromagnetic force. In economics, as seen below, a version of the oscillator can also be used to simulate the dynamics of supply and demand, and is a staple of quantum finance.

What carries over in a more general sense is the idea of representing a quantum system as a collection of particles, that can be added, removed, or translated through the use of operators. Indeed, another interpretation of the quantum model – known as the Fock space representation – is to see the harmonic oscillator as representing, not a single particle, but a collection of n fictitious particles each with energy $\hbar\omega$. In this picture, the creation and annihilation operators are seen as adding and removing these particles. The ground or vacuum state $|0\rangle$ has no particles, $|1\rangle$ has a single particle, $|2\rangle$ has two, and so on. This method, known as second quantization, underpins the quantum field theory of relativistic particles, used for example to represent systems of bosons. As seen in the next section it can also be applied to things like assets, where here n refers to the number of units held.

The other thing which carries over to economics is the different nature of classical and quantum systems. While the classical harmonic oscillator is just a weight bouncing around on a spring, where quantities such as position, momentum, and energy can be precisely calculated, the quantum version is better described in terms of potentiality. We can only calculate the probability that a measurement will yield a particular result; and the complexity of quantum behaviour means that even this can only be easily done for relatively simple systems. In economics, this puts a strong limit on how much can be gained from using reductionist methods.

6. The quantum market

In the examples above we have seen that a person's cognitive state, or the state of a quantum harmonic oscillator, can be simulated as a member of a Hilbert space. Furthermore, single particles that are in superposed states can be viewed, in a dual sense, as a collection of fictitious particles in single states. We can do something similar for the economy as a whole, and model it as a collection of interacting particles in a Hilbert space. As a starting point, we will consider a simplified financial market. I will follow here the approach described by the late Rutgers theoretical physicist Martin Schaden in a 2002 paper on quantum finance, see that paper for details and applications.³⁶

Suppose that the market consists of a collection of agents (investors) $j = 1, 2, \dots, J$ who buy and sell assets of types $i = 1, 2, \dots, I$. Each agent holds cash (or debt) x^j . The market can be represented as a Hilbert space H , with the basis

$$B := \{|x^j, \{n_i^j(s) \geq 0, i = 1, \dots, I\}, j = 1, \dots, J\}\}.$$

Here $n_i^j(s)$ is the number of assets i with a price of s dollars that are held by investor j .

An individual basis state represents a market where the price of every security, and the cash position of each agent, is known precisely. The basis states are orthogonal in the sense that if the market is in the state $|m\rangle$ then it cannot be in a different state $|n\rangle$, so if $m \neq n$ then the inner product $\langle m|n\rangle = 0$. In general the market state (wave function) M is never known this accurately and is instead represented by the linear superposition of basis states $|n\rangle$ in B :

$$|M\rangle = \sum_n A_n |n\rangle$$

where the A_n are complex numbers, and $w_n = |A_n|^2$ is the probability that the market is in the state $|n\rangle$.

The phases of the A_n are left unspecified at this stage, but are key to understanding effects such as interference. As in quantum physics, these effects are seen more easily when considering individual transactions. The propensities of each agent to buy or sell an asset can themselves be modelled as quantum phenomena, which as already discussed experience interference effects, and these can interact to affect the market as a whole. We return to this below.

³⁶ Schaden, M. (2002), 'Quantum finance', *Physica A* 316(1), pp. 511-538.

If we define the ground state $|0\rangle$ to be a market where agents hold no assets including cash, then we can build up a real market by transferring cash and assets to agents. The approach is the same as that used in many-body quantum mechanics to simulate the behaviour of a collection of bosons, so shares are added or removed from an agent's account by the use of the creation operator $\hat{a}_i^{\dagger j}(s)$ and the annihilation operator $\hat{a}_i^j(s)$. Money creation is handled using a translation operator of the form

$$\hat{c}^{\dagger j}(s) = \exp\left(-s \frac{\partial}{\partial x^j}\right)$$

which increases the amount of cash held by agent j by s currency units. Similarly the Hermitian conjugate operator $\hat{c}^j(s) = \hat{c}^{\dagger j}(-s)$ lowers the cash holding of agent j by the amount s .

While it might not be obvious from these dry equations, and we haven't considered factors such as the creation of money objects through the issuance of debt, money still has a very special (but usually understated) role in the quantum model. Unlike other assets, it has a stable defined price. Without money, it is impossible to assign a price to other assets in the first place. The fact that these assets have indeterminate value is what gives money its dualistic properties, combining as it does stable numbers and unstable values. While it isn't possible for an asset to have a negative price, an agent can have a negative amount of money. Finally, money is often created in the first place through loans, which lead to entanglement as discussed below.

The buying and selling of one unit of an asset by agent j at price s is represented by the creation and annihilation operators in combination with cash transfers which reflect the exchange of money:

$$\begin{aligned}\hat{b}_i^{\dagger j}(s) &= \hat{a}_i^{\dagger j}(s)\hat{c}^j(s), \\ \hat{b}_i^j(s) &= \hat{a}_i^j(s)\hat{c}^{\dagger j}(s).\end{aligned}$$

We can build up an arbitrary market state from the vacuum state by using these operators to successively transfer cash and securities to each agent. To study how the market wave function evolves with time, we write

$$|M\rangle_t = \hat{U}(t, t_0)|M\rangle_{t_0}$$

where $\hat{U}(t, t_0)$ is a unitary linear operator. The dynamical behaviour of the system is driven by a Hamiltonian $\hat{H}(t)$, which again satisfies the Schrödinger equation

$$i \frac{\partial}{\partial t} |M\rangle_t = \hat{H}(t) |M\rangle_t.$$

It is then possible to develop Hamiltonians for things like cash flow, the trading of securities, and so on (although the mathematics is usually more complicated than for something like the harmonic oscillator). As shown by Schaden and other researchers, these in turn can be used to derive statistical properties of markets.

The variables of the system can be loosely interpreted in terms of physical analogies. The price s of an asset (or more correctly its logarithm) is like position. As in physics, there is an uncertainty relation involving asset price, and the momentum of the price change. The creation of money or assets adds energy (as measured by the Hamiltonian) to the total energy of the system. The same techniques used to study many-body quantum systems can then be applied to make predictions about market behaviour, either in closed form or by explicitly modelling each agent.

As a simple example of a Hamiltonian in finance, consider the case of a savings instrument containing an initial amount of cash x_0 which accumulates at an interest rate r . The classical Hamiltonian for this system is

$$H = rxq$$

where (in classical notation) q is the conjugate variable of x .³⁷ We then have

$$\begin{aligned} \frac{dx}{dt} &= \frac{\partial H}{\partial q} = rx \\ \frac{dq}{dt} &= -\frac{\partial H}{\partial x} = -rq. \end{aligned}$$

Solving then gives

$$\begin{aligned} x &= x_0 e^{rt} \\ q &= q_0 e^{-rt} \end{aligned}$$

which implies that the Hamiltonian is constant in time:

$$H = rxq = rx_0 e^{rt} q_0 e^{-rt} = rx_0 q_0.$$

Note that changing q_0 doesn't affect the result for x , so we can set $q_0 = 1$ which means that $q = e^{-rt}$ is the value of one unit of currency discounted to time $t = 0$.

³⁷ See e.g. Bensoussan, A., Chutani, A. & Sethi, S. (2009), 'Optimal Cash Management under Uncertainty', *Operations Research Letters* 37:425-429.

In analogy with a physical system, the amount of money x can be interpreted as having a dimension of length L . The momentum $q = m\dot{x}$ has units MLT^{-1} (mass times length over time), a force acting on the momentum $s = \dot{q} = m\ddot{x}$ has units MLT^{-2} , and the work performed by the force has units ML^2T^{-2} . Because the system is blowing up in size (so becoming less dense) with no inputs of energy, the inertial mass term is not constant but decreases exponentially, with solution $m = m_0 e^{-2rt}$ where $m_0 = \frac{q_0}{rx_0}$. As in a nuclear reactor, the mass is being converted into another form of energy.

To quantize the system, we again replace the Hamiltonian H and classical variables x and p with operators. Because the Hamiltonian must be Hermitian, we need to write it in a symmetric form as

$$\hat{H} = \frac{r}{2}(\hat{x}\hat{q} + \hat{q}\hat{x}).$$

Standard techniques can then be used to show that the probability distribution of the cash holdings matches that expected from the classical case (as Schaden notes, the quantum approach only comes into its own when future returns are uncertain). One can draw an analogy with the Hamiltonian of a multi-boson system $\hat{H} = \hbar\omega\left(\hat{N} + \frac{1}{2}\right)$. The interest rate r , which like ω has units of inverse time, plays the role of frequency (another way to see it is as frequency of a fixed payment), while the initial investment plays the role of the number operator \hat{N} (plus the $\frac{1}{2}$ contribution of the ground state). In the case of a single cash transfer of a quantity s at time $t = t_0$, the Hamiltonian becomes $\hat{H}(t) = s\delta(t - t_0)\hat{q}(t)$ where the delta function $\delta(t - t_0)$ has the value 1 at $t = t_0$ and 0 at other times.

The cash flow model treats the account as a black box which magically produces money at a fixed rate r . There are no inputs or outputs, which is why the Hamiltonian remains constant even as the nominal amount of money increases indefinitely. While such isolated systems do not exist in reality, the simple model – when coupled with the idea of quantum money creation – is instructive about how inflation occurs in something like a housing market. As emphasised in the book, money is created by private banks every time they issue a mortgage. If we assume mortgage lending continues at a steady rate, then the money supply will grow at some rate r (in Figure 3 of *Quantum Economics* the Canadian money supply grows at an annual rate of about 6.5 percent, so $r = 0.065$). If this money is then used to bid up the price

of houses, then house price growth will track money supply growth, even if the real value of homes remains unchanged.

An important difference between cash and a security is that while money is a conserved quantity during transactions, a security once bought evolves into a superposition of states, each of different prices, with amplitudes specifying the probability of selling at that price. As an example, suppose that a particular investor initially has no shares in a particular company, and then acquires one share at time 0 for a price s_0 .³⁸ By making a number of simplifying assumptions, and some rather involved computations, Schaden shows that the probability of selling the stock a time T later for price s follows a lognormal distribution which depends on the expected return and volatility of the stock.³⁹ This is a well-known empirical result, that can be derived from standard stochastic approaches, so serves primarily as a consistency check. However it only holds for intermediate time scales of a month or more, and again assumes that the market is near equilibrium. The quantum approach helps to explain how this model breaks down at shorter time scales, or for assets which are infrequently traded.

To summarise this section, a market can be represented as a Hilbert space, in which the price of an asset is known precisely only at the time of a transaction. Ownership and context are important, so an asset purchased by one person at one price is distinct from the same asset purchased by another person at a different price. As in quantum cognition, the act of measuring an asset's price – in this case by buying or selling – has an effect on the price. By constructing an appropriate Hamiltonian equation, we can study the dynamics of market evolution. As in physics, the complexity of the system means that macro-level behaviour is often characterised by emergent properties that cannot be reduced to some lower level. Again this differs from the classical approach which assumes assets have a certain inherent value independent of context; money does not play an important role, other than as a metric; and calculations can be based on micro-foundations of individual utility optimisation.

³⁸ The initial state $|M_0\rangle$ can be written $|M_0\rangle = \hat{b}^\dagger(s_0)|\tilde{M}_0\rangle$ where $\hat{b}(s_0)|\tilde{M}_0\rangle = 0$. Here the indices for other stocks and investors have been repressed for clarity, and \tilde{M}_0 is a state where the investor has no shares in the company (which is why the annihilation operator yields 0). At time T , the state evolves to $|M_T\rangle = \hat{U}(t, t_0)|M_0\rangle$. The probability that the investor can sell the single share at a price s can be computed by looking at the product $\langle \tilde{M}_T | \hat{b}(s) | M_T \rangle$, where \tilde{M}_T is again a state that is annihilated by $\hat{b}(s)$.

³⁹ The formula is $P_T(s|s_0) = \frac{1}{s\sigma\sqrt{2\pi T}} \exp\left[-\frac{(\ln(\frac{s}{s_0}) - \mu T)^2}{2\sigma^2 T}\right]$ where μ is expected return and σ is volatility.

Like quantum cognition, quantum finance has become a sizeable area of research, with many papers showing empirical results. If markets are assumed to be large and nearly efficient, then the results do generally approximate those produced by the classical approach. (Indeed, researchers have so far largely tended to respect classical assumptions such as efficiency, in an attempt to replicate known results.) However quantum effects become more important for markets that are thinly traded, and the quantum approach can also be used to describe markets driven by investor sentiment, where there is a significant degree of entanglement between market participants.

While quantum finance concentrates on the specialised case of financial markets, and is used for studying the properties of assets such as stocks or bonds, the same methodology can in principle be extended to describe markets in general, and form the basis of a mathematical description of the quantum economy. Again, money has a special role as an asset with a fixed price, and the price of everything else is indeterminate until measured through monetary transactions.

7. Supply and demand

While the operator approach is useful for building up a quantum model of markets, we turn now to the more basic question of the relationship between supply and demand, which depends on cognitive decisions to buy or sell.⁴⁰ The standard interpretation, known as the law of supply and demand, is traditionally illustrated using versions of a graph first published in an 1870 essay by Fleeming Jenkin. It has since become the most famous figure in economics, and is taught at every undergraduate economics class. The figure shows two intersecting curves or lines, which describe how supply and demand are related to price. When price is low, supply is low as well, because producers have little incentive to enter the market; but when price is high, supply also increases. Conversely, demand is lower at high prices because fewer consumers are willing to pay that much. The point where the two lines cross gives the unique price at which supply and demand are in perfect balance, and is therefore a pictorial representation of Adam Smith's invisible hand.

⁴⁰ See the discussion paper "A Quantum Model of Supply and Demand" available at https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3376652

The law of supply and demand not only plays an important role in many economic models, but also justifies the widespread assumption in economics that prices are drawn to a stable equilibrium. However there are a number of basic problems with it. One is that it is generally impossible to measure supply or demand curves, because all we have is transactions which involve both quantities. The parameters are therefore underdetermined. Another problem is that the law is deterministic, while economic interactions are intrinsically probabilistic (or indeterminate). The law by itself also gives little sense of underlying dynamics (according to the efficient market hypothesis, equilibrium is achieved instantly). Finally, the law assumes continuity, but goods are sold in discrete amounts, and financial transactions are inherently discontinuous.

These issues can be addressed via the adoption of a quantum formalism, which is explicitly designed to handle systems that are discrete, indeterminate, and dynamic. According to quantum finance, asset prices are indeterminate until measured through transactions, so can be modeled using wave functions that collapse to a certain price when measured. Quantum cognition, meanwhile, treats mental states as indeterminate until measured through decisions. These theories come together naturally in the question of supply and demand, which involves decisions about financial transactions.

This section applies the quantum methodology to a simple but illustrative case of supply and demand, that can be extended to a variety of situations. The probabilistic approach is similar to that of (Kondratenko, 2015), who also argues for a quantum link, however the section derives dynamic equations that are interpreted in terms of entropic forces; uses these to generate equations for an oscillator model; draws an explicit connection with recent research in quantum finance, where a similar quantum oscillator model has been used to model asset price changes; and motivates the technique's application in other areas of economics.

Case with single buyer and seller

As a starting point, first consider the case of a single buyer and seller, who are negotiating a transaction involving a certain good (say a stock, or a house). The buyer might have a certain bid price μ_b in mind, while the seller has an offer price μ_o . Because price is a relative quantity, we will treat it as a logarithmic variable. Since it generally holds that $\mu_b < \mu_o$ there will be no transaction unless at least one party shows some flexibility. It is therefore

necessary to broaden the constraints, so instead of having a central fixed price, each participant is willing to consider a range of prices, with the propensity to sell or purchase at each price described by a function. The use of the term “propensity” here is similar to that in stochastic chemical kinetics, where it refers to the probability of a molecular reaction occurring in a certain time (Lecca, 2013). The situation is shown graphically in Figure 1, where $P_o(x)$ is the offer propensity function, and $P_b(x)$ the bid propensity function. Both functions are assumed to be normal (Gaussian), with standard deviations σ_o and σ_b . The case for the common scenario where the sales price is fixed over a trading period would be modeled by setting $\sigma_o = 0$ so P_o is a delta function.

This assumption of normally distributed prices may seem a little strange, since it implies that buyers will not purchase items that seem too cheap, and sellers will be reluctant to sell above a certain price. One way to think of these curves is as a kind of schedule, where the seller and buyer mentally partition their offers and bids, with a peak at a central price which they consider to be ideal but not too unrealistic, and in a manner that is constrained by the condition that the integral of the function equals 1. Viewed this way, it would not make sense for a buyer to commit to buy only at a very low price, since they would then have to decline any reasonable offer outside that range. Note also that transactions take place in the middle ground between the mean bid and offer prices, so what matters is the behavior of the propensity functions over this range.

If we assume independence, then the joint propensity function, which describes the joint probability of a transaction actually occurring at a particular price, is the product $P_t(x) = P_o(x)P_b(x)$, shown by the blue line in the figure. The area of this graph measures the propensity for trade. It is easily shown (Bromiley, 2018) that the product of two normal distribution curves is a scaled normal curve, with mean and standard deviation

$$\mu = \frac{\sigma_b^2 \mu_o + \sigma_o^2 \mu_b}{\sigma_o^2 + \sigma_b^2}$$

$$\sigma = \frac{\sigma_o \sigma_b}{\sqrt{\sigma_o^2 + \sigma_b^2}}$$

The scaling factor α is itself a normal distribution of the form

$$\alpha = \frac{1}{\sqrt{2\pi\sigma_t^2}} e^{-\frac{\mu_t^2}{2\sigma_t^2}}$$

where $\mu_t = \mu_o - \mu_b$ is the spread, and the standard deviation $\sigma_t = \sqrt{\sigma_o^2 + \sigma_b^2}$ is a measure of price flexibility (Bromiley, 2018). The main parameters and equations are also summarized in the Appendix of the discussion paper.⁴¹

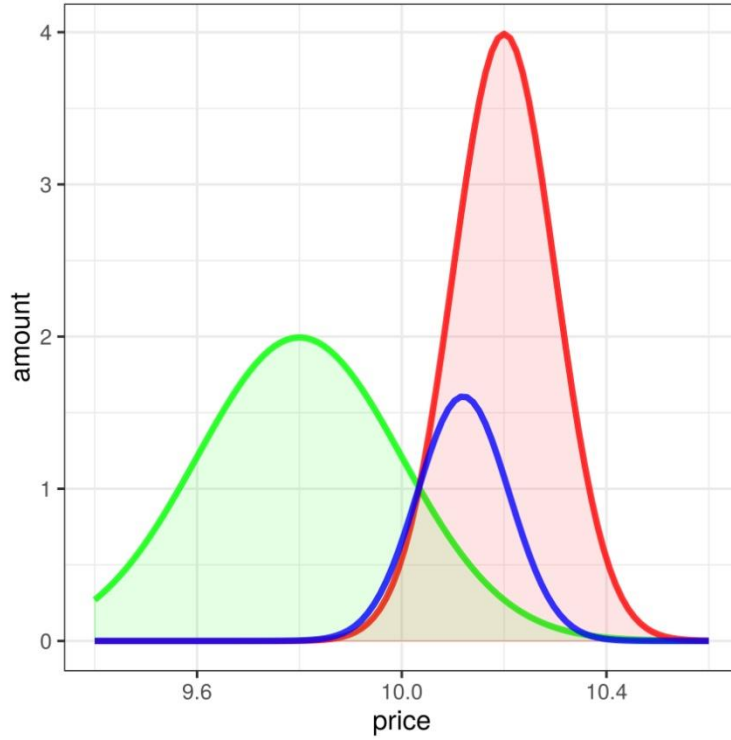


Figure 7.1. Plot showing the buyer's propensity function (green) and the seller's propensity function (red). The joint propensity for a transaction occurring is the product of these functions, shown by the blue line. Price is treated as a logarithmic variable.

For the case of a financial market, price quotes in the order book often come from market makers. The expected profit over a trading cycle for a market maker will depend on the operating spread, which represents the profit per transaction, multiplied by the amount traded. The propensity for trade scales with α so if we assume the operating spread scales with μ_t , then the profit scales with the product $\mu_t \alpha$ which has a maximum when the spread satisfies $\mu_t = \sigma_t$. If we further assume that market makers adjust the spread in this way in order to maximize profit, then setting this value for μ_t in the expression for the propensity for trade α gives $\alpha \propto \exp(-H)$ where $H = \log(2\pi e \sigma_t^2)/2$ is the differential entropy of the normal distribution (Norwich, 1993). Market negotiations which align the expectations of the buyer and seller will also tend to reduce σ_t and therefore minimize the entropy, which in

⁴¹ see https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3376652

information theoretic terms is equivalent to minimizing the missing information about the system (Williams, 1980).

Multiple agents

So far we have only considered the case of a single buyer and seller, who are negotiating the price of a single item, but the same methodology carries over easily to the case with multiple units and agents. The bid functions and offer functions are aggregated to give a total bid function over all buyers, and a total offer function over all sellers, measured in numbers of units. In an agent-based model this would be performed by summing the propensity functions directly. If for simplicity we assume that the bid propensity functions all share the same mean and standard deviation, and likewise for the offer functions, then the effect is to simply scale the propensity functions by the numbers of sellers and buyers respectively.

The expected trading volume, expressed as a rate, is then given by $V = N_o N_b r \alpha$. This is the same as mass action kinetics in chemistry, where a reaction between two chemical species in solution occurs at a rate which is proportional to the concentrations of the reactants, and their chemical affinity, but depends also on factors such as the temperature. Here the propensity for trade α is adjusted by a rate parameter r which accounts for the exact structure of the market and the degree and nature of the interactions between buyers and sellers. Note that as in chemistry (Brogioli, 2013), this equation serves as a useful first-order model but may need to be modified under certain conditions (or agents can be modeled individually). In a stochastic model the number of transactions over a trading cycle of duration τ follows a Poisson distribution with mean $\lambda = V\tau$.

This population model also gives a different perspective on the propensity functions. The red line in Figure 2 shows the cumulative number of units $C_o(x)$ sold to an individual buyer, assuming that the buyer starts with the lowest available offer price and works their way up to the price x . This curve is given by the seller cumulative propensity function multiplied by the number of sellers:

$$C_o(x) = N_o \int_{-\infty}^x P_o(x) dx.$$

The green line shows the number of units $C_b(x)$ sold by an individual seller, assuming that the seller starts with the highest available bid price and works their way down to the price x , which is given by

$$C_b(x) = N_b - N_b \int_{-\infty}^x P_b(x) dx.$$

These cumulative propensity curves are very unrealistic because they assume that single large buy and sell orders are broken into infinitesimal chunks and processed in order. In reality, transaction charges would mean that the orders would be handled in a small number of large transactions. The curves also remove any probabilistic uncertainty, because buyers and sellers are assumed to have perfect information. However they are interesting because they resemble the traditional plots of supply and demand, with the difference that the independent variable price is on the horizontal, rather than vertical, axis. If we consider cash as carrying momentum (Fischer & Braun, 2003), then a large purchase (or sale) can be viewed as a transfer of momentum which will perturb the price point. Far from being an inert medium of exchange, money is the basis of a measurement procedure which affects the system being measured (Orrell, 2017: 20).

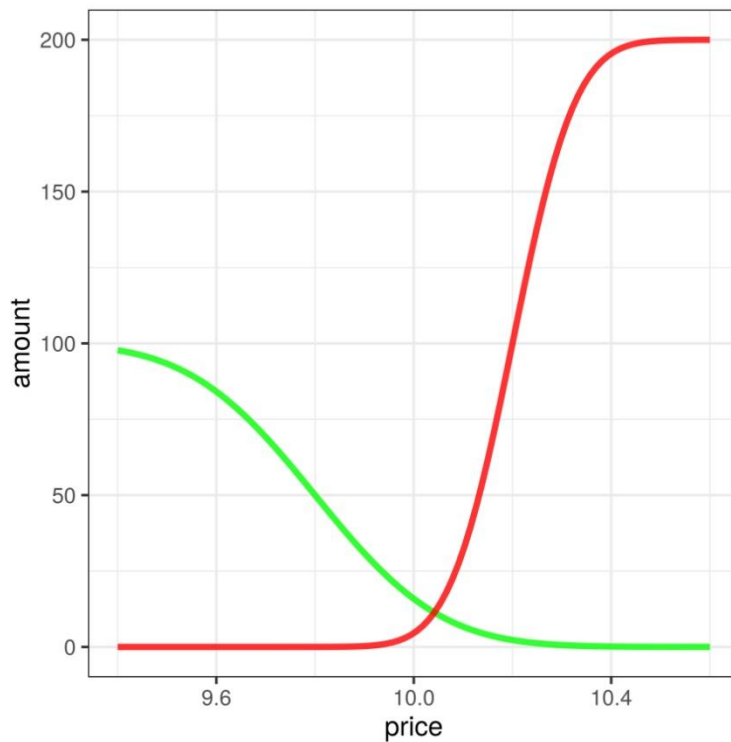


Figure 7.2. Cumulative bid/offer propensity curves, scaled by the number of participants, for the case with 100 buyers and 200 sellers. The red line shows the total number of units sold $C_o(x)$, assuming that an individual buyer starts with the lowest available price and works

their way up to the price x . The maximum for this curve is 200 which corresponds to the number of units available, assuming vendors sell a maximum of 1 unit each. The green line shows the number bought $C_b(x)$, assuming that an individual seller starts with the highest available price and works their way down to x .

We here notice a clear difference between the deterministic and probabilistic interpretations. In the former, the equilibrium price is the intersection point at which supply and demand are equal, obtained with a probability 1, while in the latter, the expected price is normally distributed. What counts in the probabilistic picture is not just the number of buyers or sellers, but their flexibility in negotiating prices, as expressed by the inverse of the variance. As discussed further below, the model can be generalized to simulate group influences where suppliers collectively decide to change their price ranges.

This simple model assumes that buyers and sellers in the population are homogeneous in the sense that they share the same offer and bid functions. Even without this assumption, it should often be possible to approximate the total offer function using a normal distribution. Also, while we have considered normal distributions here because of their mathematical convenience, one could consider different shapes for the propensity functions. The main thing is that the product of these functions, in the region around the price-point of interest, should be approximated by a normal curve, which is the case if the buyer and seller forces defined below are locally linear. In general, it seems reasonable to suppose that transactions will occur over a limited range and can be approximated by the kind of model described here.

The entropic oscillator

The bid and offer propensity functions in Figure 1 can be viewed as representing the mental state of the buyer/seller. As shown by cognitive psychology, decisions contain a random component, so should be modeled as probabilistic processes (Bussemeyer & Bruza, 2012). However we can also think of these curves as describing a kind of force. To motivate the treatment, suppose that the current price x is higher than the buyer's central price μ_b . The probability to purchase is then given by the propensity function $P_b(x)$. The resistance to changing to some nearby price $x + \Delta x$ will depend on the change in propensity conditional on (or relative to) the current propensity. This is equal to the slope of the propensity, divided

by the current propensity, or $P_b'(x)/P_b(x)$. We therefore define the supply and demand forces as

$$F_o(x) = -\gamma \frac{P_o'(x)}{P_o(x)} = \frac{\gamma(x - \mu_o)}{\sigma_o^2} = k_o(x - \mu_o)$$

$$F_b(x) = \gamma \frac{P_b'(x)}{P_b(x)} = \frac{-\gamma(x - \mu_b)}{\sigma_b^2} = -k_b(x - \mu_b)$$

where $k_o = \gamma/\sigma_o^2$ and $k_b = \gamma/\sigma_b^2$ are force constants, and γ is a scaling parameter with units of energy.⁴² The demand force slopes downwards, because there is resistance to increasing price, while the supply force slopes upwards. The forces therefore represent the mental desire for the buyer or seller to adjust the price to their own preferred level. Note that, because the propensity functions are chosen to be normal curves, the corresponding forces are linear in price. They can therefore be viewed as a first-order approximation to the dynamics in the region of the central equilibrium point. As seen in the Appendix, these forces are the cognitive version of entropic forces which reflect the tendency of a thermodynamic system to maximize entropy by evolving to states that are statistically more probable (with the difference that they act in the opposite direction, so decrease the entropy).

We can similarly define the transaction force as the entropic force generated by the joint probability, which is just the sum of the buyer and seller forces:

$$F_t(x) = \gamma \frac{P_t'(x)}{P_t(x)} = \gamma \frac{P_o(x)P_b'(x) + P_o'(x)P_b(x)}{P_o(x)P_b(x)} = F_o(x) + F_b(x).$$

The point at which the probability of a transaction is highest can be found by setting the derivative of the joint propensity function to zero, so

$$P_t'(x) = P_o'(x)P_b(x) + P_o(x)P_b'(x) = 0$$

or

$$F_o(x) = -F_b(x)$$

which occurs at the price

$$\mu = \frac{k_o\mu_o + k_b\mu_b}{k_o + k_b}.$$

The equilibrium price is therefore the point where the supply and demand forces are in balance and $F_t(x) = 0$, as expected.

⁴² (Kondratenko, 2015: 137-138) proposes similar force terms, without a corresponding expression for mass, on the basis that these terms cancel at equilibrium.

The existence of a restoring force is consistent with the idea that market sentiment tends to oscillate over time, alternating between periods of e.g. greed and fear. To picture the dynamics, we can imagine the force $F_t(x)$ acting on a mass $m = m_o + m_b$, where m_o and m_b represent the resistance to change of the seller and buyer respectively, and these masses are joined together as shown in Figure 3. The equation of motion for this coupled system (diagram C in the figure) is then $m\ddot{x} = -k(x - \mu)$ where $k = k_o + k_b$. This has the oscillatory solution $x = \mu + A \cos(\omega t + \varphi)$ where A is the amplitude, $\omega = \sqrt{k/m}$ is the frequency of oscillation, and the phase φ depends on the starting point.

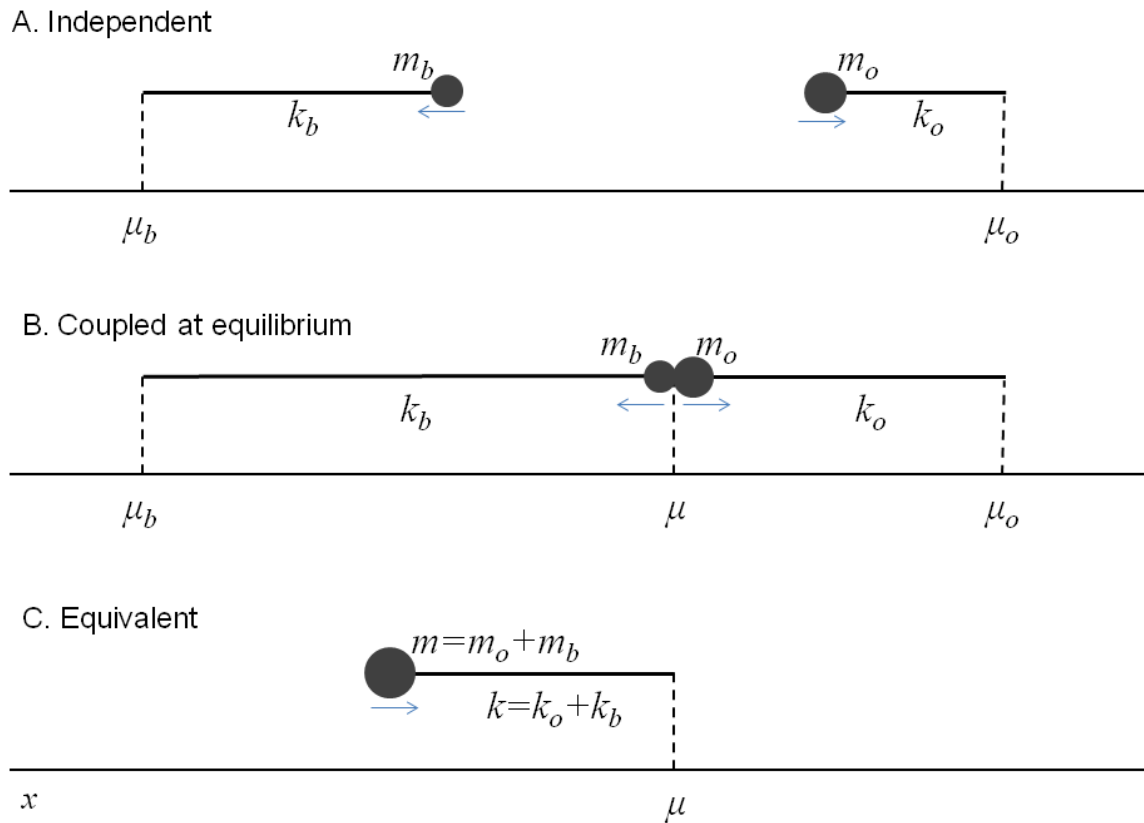


Figure 7.3. In diagram A the seller and buyer are represented by the masses m_o and m_b which oscillate independently around their central points with spring constants k_o and k_b . The arrows show the direction of force. Diagram B shows the coupled system where the two masses are attached, and are at their equilibrium point μ . This is equivalent to the oscillator in diagram C with mass $m = m_o + m_b$ and spring constant $k = k_o + k_b$.

While such a force would express the restoring tendency towards a central price, it again is deterministic rather than probabilistic. Also, unless additional damping terms are added, the

price would simply bounce back and forth between two extremes that would depend on the initial conditions. The probability distribution of prices is given by the equation

$$P(x, A) = \frac{1}{\pi \sqrt{A^2 - (x - \mu)^2}}$$

which as shown in Figure 4 below is highest at the extremes (where the rate of change is slowest) and lowest in the midpoint (where it is fastest), which is inconsistent with the probabilistic picture in Figure 1.

A more realistic approach would therefore be to assume the oscillator is driven by random noise. This results in an Ornstein-Uhlenbeck process which is a mean-reverting random walk given by the stochastic differential equation

$$dx = -\theta kx dt + \sqrt{2D} dW.$$

Here dW is a Wiener process, D is the diffusion coefficient, and θ is the mobility which measures the drift velocity induced by a given force (Titievsky, 2005). The probability density function P then satisfies the Fokker-Planck equation

$$\frac{\partial P}{\partial t} = \theta k \frac{\partial}{\partial x} (xP) + D \frac{\partial^2 P}{\partial x^2}.$$

The steady state solution $P(x)$ is a Gaussian with standard deviation $\sigma = \sqrt{D/k}$. Excited states relax back to this steady state due to dissipation. In physics, if we assume that the system is perturbed by thermal noise, then according to the Einstein relation we have $D = \theta k_B T$, where k_B is the Boltzmann constant and T is temperature. In the quantum model developed below the force constant satisfies $k = \gamma/\sigma^2$ from which it follows that $\gamma = D = \theta k_B T$. For $\theta = 1$ this is the same relationship arrived at by interpreting the supply and demand forces as entropic forces, see Appendix of the paper.

The quantum harmonic oscillator

While it is certainly reasonable to model the system as a stochastic differential equation in this way, an alternative approach is to shift to a quantum framework, which offers a natural way to handle its indeterminate, dynamic properties. For example, while the stochastic approach assumes that price has a well-defined value at each time, the quantum model acknowledges that prices, and indeed the mental states of buyers and seller, are indeterminate until measured through a transaction, and this measurement process affects the price. As

mentioned in (Segal & Segal, 1998: 4072), an obvious feature of financial markets is that it is impossible to observe both an asset's price, and its instantaneous rate of change: this "lack of simultaneous observability appears capable of precise mathematical formulation only in quantum terms." More generally, as mentioned in the introduction, the information flows that make up economic transactions do not behave in a classical fashion and are often better suited to a quantum approach.

In the quantum formalism, the price of an asset is represented by a wave function that collapses to a particular value when measured through a transaction, just as the wave function for a particle's position collapses down to a single number when measured. We can move to the quantum framework by "quantizing" the classical equations. The quantum version of the spring equation is the quantum harmonic oscillator, which is restricted to a discrete set of energy levels. The ground state is described by the wave function

$$\psi_E(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}(x - \mu)^2\right).$$

The corresponding probability distribution for x is a normal distribution with mean μ and standard deviation

$$\sigma = \sqrt{\frac{\hbar}{2m\omega}}.$$

From the Schrödinger equation, the time evolution of an oscillator in the ground state is given by the complex wave function

$$\psi_E(x, t) = \exp\left(-\frac{i\omega t}{2}\right) \psi_E(x)$$

which rotates around the imaginary axis with an angular frequency $\omega/2$. As the energy increases, the probability distribution peaks at the extremes, as with the classical case in Figure 4, in accordance with Bohr's correspondence principle (Rae, 2008: 40). In practice only low energy states will be used here, as discussed in Section 6.

To see how this applies to supply and demand, we identify the probability distributions as the ground states of quantum oscillators. For the buyer or seller, the oscillator can represent a kind of mental oscillation over prices, while for the transaction price it represents an oscillation between the buyer's preferred price and that of the seller. As discussed below the parameter \hbar in this model will determine the transition between energy levels, while ω represents a characteristic frequency. It is then easily shown (see Appendix) that the

equations for the mean and standard deviation of the quantum oscillator for the coupled supply/demand system are the same as the equations for the joint propensity function from Section 2. Mass terms for the buyer and seller scale inversely with the variance of the propensity functions, so are a measure of price flexibility. The corresponding price distribution is shown in Figure 4, where it is assumed that a transaction has taken place (so the total probability is 1).

This equivalence between the models does not rely on the fitting of any parameters. The sole additional assumption is that demand and supply forces scale in a consistent way with variance of the propensity functions. A consequence is that the frequencies of the buyer and seller in the transaction process are the same. In the classical model (Figure 3) this was enforced by physically joining their corresponding masses, as is necessary if they are to represent a single price.

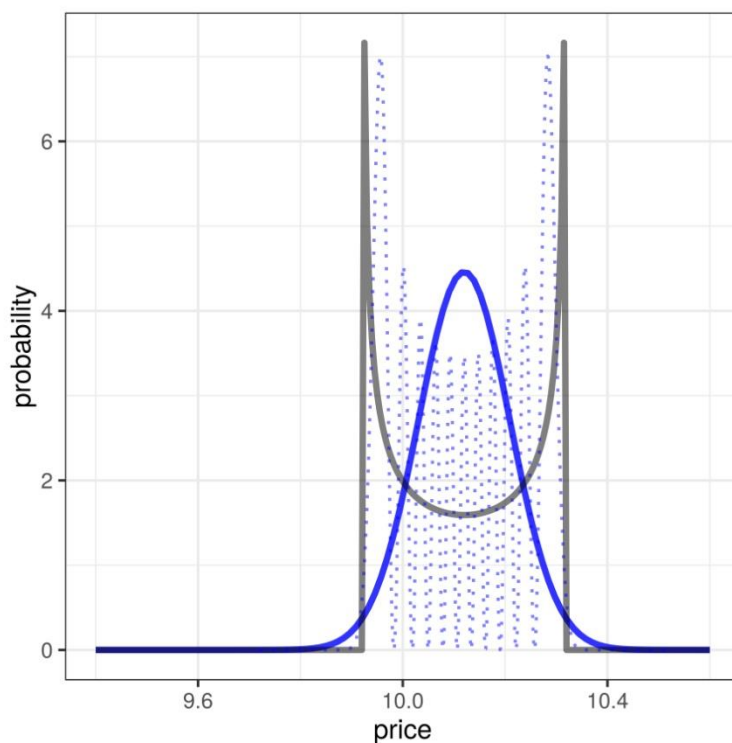


Figure 7.4. Probability distributions for position for the classical harmonic oscillator (grey line) at high energy, and the quantum model in its ground state (blue line) and in the tenth excited eigenstate (dotted). In the classical oscillator model, the probability is highest at the extremes rather than at the center, and the range – which here equals the bid/offer spread – depends on the initial conditions. The quantum oscillator matches the probability distribution of price for transactions at low energy, and converges when smoothed to the classical

distribution as energy increases. The tenth eigenstate is shown for illustrative purposes, only the lower-energy states are typically used.

As discussed in the final section, the parameters \hbar and ω serve as a parsimonious way to fit the higher energy level states that are characteristic of observed data. Comparison with the entropic version (see Appendix) shows that $\gamma = \hbar\omega/2 = \theta k_B T$ where the mobility factor θ was set to 1. In physics, both \hbar and k_B are well-defined constants, with one quantizing mechanical action and the other quantizing entropy (Fernández de Córdoba et al., 2016). In economics they have no such set value, however the presence of the mobility factor, which depends on the details of the system, is a reminder that in economics these parameters are context-dependent, and need to be fit for a particular model. Note also that quantizing the system changes the meaning of the parameters (instead of a scaled temperature, there is a scaled frequency) but does not increase their number, which is important from a modeling perspective.

In the classical picture, with price modeled by a classical oscillator (which can be viewed as representing a kind of dynamic bargaining process), one would represent a price negotiation by adding energy to the system in order to induce an oscillation. In the quantum picture, we can similarly add an amount of energy E_d using a displacement operator (for example, if the system is initially in the ground state, then a displacement of 2σ raises the energy by $E_d = \hbar\omega$). Again, this can be viewed as the result of a negotiation process, where the forces exerted by the buyer and seller shift and adjust in response to each other. Interactions between groups of buyers or sellers could have similar effects. The probability of observing the system in a particular state then follows a Poisson distribution with mean $\lambda = E_d/(\hbar\omega)$.

A number of authors have developed models of financial trading based on the operator approach, as in quantum field theory where creation and annihilation operators are used to model the behavior of particles (Schaden, 2002; Bagarello, 2006; Haven et al., 2017; Khrennikova & Patra, 2019). For example in the model of (Gonçalves & Gonçalves, 2008) the number of buyers and sellers is represented by a number operator which counts the participants. At the start of each trading cycle, the system is in a ground state which is then perturbed by a displacement operator. This puts the system into a so-called coherent state where the number of participants follows a Poisson distribution. While a discussion of the

operator approach is beyond the scope of this paper, it does suggest the following possible interpretation, which is to treat the supply/demand system at the start of a trading cycle as a quantum oscillator in its ground state. When a negotiation begins, the effect is to perturb the system with an increase in energy of $E_d = \hbar\omega\lambda$ units. In this picture, the bringing of money to the table therefore acts as a kind of financial kick to the system. The result is a coherent state, which can be viewed as a quantum version of a classical oscillating state. The probability density is Gaussian but oscillates around the mean, and the energy level when measured follows a Poisson distribution with mean λ , so corresponds to the number of transactions in the probabilistic model.

To summarize, the state of the system is being modeled as a quantum harmonic oscillator whose properties can all be derived from the probability distributions for the buyer and seller, as measured in a trading context. The energy of the oscillator, and therefore the probability of transactions occurring during a trading cycle, reflects both price spread and price flexibility. The quantum model can be seen as mediating between two classical models: the ground state corresponds to the normal-shaped static probabilistic model of supply and demand, while as energy increases (i.e. for excited states) the model converges to the dynamic spring model where prices oscillate around the mean, and have the highest probability of being observed at the extremes. The statistical behavior is essentially the same as that derived from a classical stochastic model, however there are a number of key differences. The system state is modeled by a complex wave function, and variables such as prices or the mental states of the buyers and sellers are treated as indeterminate until measured through a transaction, which as discussed further below has implications for things like interference effects and entanglement. The quantum model has a non-trivial ground state which reflects uncertainty rather than random noise. Also, while excited states damp out in the stochastic model, in the quantum model they persist until the wave function is collapsed through measurement. The main operating assumption is that the buyer and seller forces $F_b(x)$ and $F_o(x)$ are linear in the region of the equilibrium price. The entropic nature of these forces makes clear the connection between information exchanges, quantum behavior, and economic transactions.

Discussion

The probabilistic quantum model can be applied in a number of ways to model financial transactions. The most basic is to use it as a way to generate stochastic models of supply and demand. For example, Figure 5 shows two simulations for a simple system where the price of some good is adjusted by the seller so as to maintain a certain level of inventory (see Appendix for details). The dashed line shows the equilibrium demand level using a classical systems dynamic approach. The solid line shows a scenario where the price is set by the seller as before, but now the number of units purchased at that price follows a Poisson distribution as described in Section 3. The effect is to create stochastic noise in the price level, even when the system is unperturbed. In other words, random changes are here caused not by external events, as assumed in conventional theories such as the efficient market hypothesis (Fama, 1965), but are due to the innate uncertainty of the system. Again, this is because the quantum oscillator has a ground state with non-zero energy.

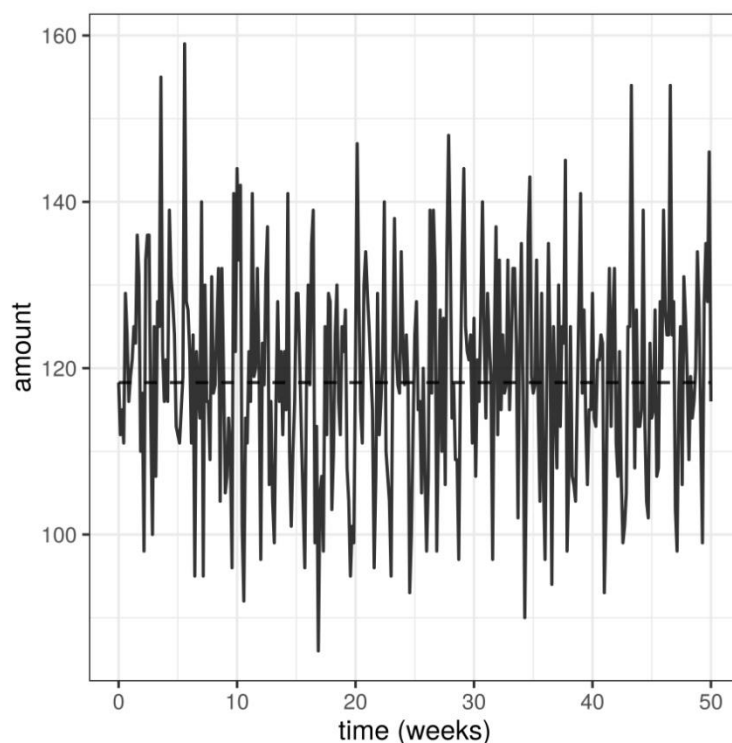


Figure 7.5. Straight dashed line shows a simulation of demand in a systems dynamics model where price is set dynamically by the seller in order to maintain inventory at four times the level of demand. The system starts at equilibrium so demand is stable. Solid line shows a simulation where price is again set dynamically by the seller, but the demand at that price is stochastic.

Such stochastic models have been widely used in areas such as systems biology, where it has been shown that certain system properties such as negative feedback actively damp stochastic variations due to a small number of molecules (Orrell & Bolouri, 2004; Ramsey et al. 2006), but in economics their use is usually limited to assessing the effects of random external shocks rather than internal dynamics. A first step therefore would be to replace deterministic supply/demand equations in conventional models with dynamic probabilistic versions. Larger models could take advantage of the computational techniques developed for systems biology models (Ramsey et al. 2005).

This type of application would only exploit the probabilistic aspect of the approach; however the most interesting features of the quantum oscillator are its quantized energy structure, and the possibility for phenomena such as interference and entanglement between multiple oscillators, which are very relevant for economics. As a simple illustrative example, Figure 6 shows a prototype quantum agent-based model where 100 buyers and 200 sellers perform transactions. The mass term of the buyers has an oscillatory component which gives a seasonal variation. In terms of quantum decision theory (Yukalov & Sornette, 2014), this variation could be attributed to a subjective attraction factor which reflects seasonal attitudes and interferes with objective calculations of utility on the part of the buyer. Seasonal behavior can also of course be produced by classical models, but the advantage of the quantum approach is that it provides a general framework for handling such effects.

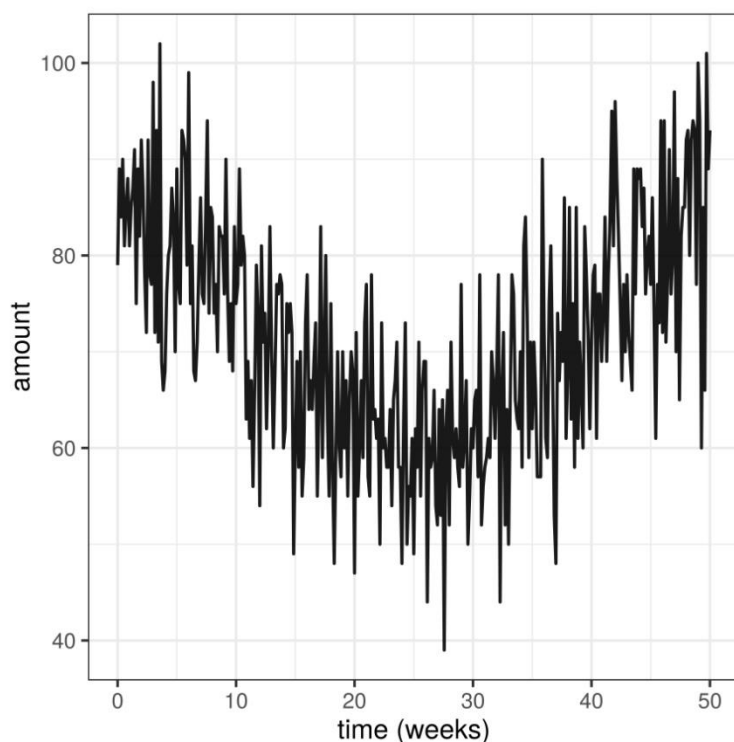


Figure 7.6. Simulations of the daily number of transactions for a model where the mass term for the buyer has a seasonal component. When buyers are less flexible (higher mass) there are fewer transactions. Price follows a similar pattern.

As mentioned in the introduction, a number of authors have previously used the quantum oscillator to model asset price changes in financial markets as oscillations in a potential well (Piotrowski & Śładkowski 2001; Meng et al. 2015; Ahn et al. 2017), with the restoring force representing reversion to the mean (as opposed to the bottom-up interpretation here in terms of probabilistic interactions between buyer and seller). The quantum Hamiltonian can be viewed as an expression of a stock's risk: the kinetic term captures the degree of price momentum, while the potential term reflects deviation from equilibrium. The mass m is seen as reflecting properties such as market capitalization that affect the rate of price adjustment, while ω is a characteristic oscillating frequency. Piotrowski et al. (2006) also derived a model of asset price changes, based on quantum game theory, that followed a Ornstein-Uhlenbeck process, and used it to obtain a formula for the price of a European call option.

As seen above, energy levels in the oscillator are quantized, with a normal ground state and higher energy levels that show more complicated distributions. Ahn et al. (2017) showed that the quantum oscillator model outperformed traditional stochastic process models for fitting

historical price changes in the Financial Times Stock Exchange (FTSE) All Share Index. The system was found to be in the ground state nearly all (about 98 percent) of the time, with the next two levels contributing the skewness and kurtosis that characterized the data. Higher levels had negligible effect. The frequency ω was interpreted as a measure of the speed of mean reversion of stock returns. This number will of course depend on the particular market and asset; for example (Balvers, Wu & Gilliland, 2000) analyzed a number of stock markets and estimated a reversion half-life of three to three and a half years.

The quantum approach is also compatible with Ising-type models from statistical mechanics which have been used to simulate stock market dynamics (Bouchaud, 2009). In physics, the Ising model was initially developed to simulate ferromagnetic materials, where the magnetic dipole moments of atomic spins can be in one of two states (+1 or -1). When an external magnetic field is applied, interactions between atoms lead to phase transitions between a random state and ones in which spins are aligned. The same idea can be applied to simulate contagion in the stock market, where market participants collectively change their stance towards asset valuation. For example Gusev et al. (2015) created an empirically-fitted model where prices oscillate in a potential well which is determined in part by the propagation of news and opinions. While they used a classical version of the Ising model, a quantum version would give similar results, though again with the feature of a ground state where fluctuations occur even in the absence of new information.

Finally, financial markets are characterized by entanglement of two sorts. The first is through social factors such as culture or news, the second (and more direct) is through the use of financial instruments such as loans and derivatives. As discussed in (Orrell, 2018; 2018a) a loan agreement can be modeled as an entangled system, where the borrower's mental state to pay or default is a quantum state which is entangled via the loan with that of the creditor (so default immediately affects the status of the loan even if the creditor doesn't find out immediately). Entangled oscillators are a staple of quantum physics and some of the techniques could carry over into economics. The oscillator model of supply and demand could for example be incorporated in quantum agent-based models where decisions to buy or sell are viewed as the outcome of quantum dynamic processes, that are susceptible to entanglement through social influences but also through the financial system itself. The development of such a model is a project for future work.

To sum up, the quantum approach provides a natural framework for modeling supply and demand. The main parameters are measures of preferred prices and flexibility, which comprise a minimal description of buyer/seller behavior. To summarize, the main conclusions are:

- The propensity for buyers and sellers to take part in a transaction can be modeled as a joint propensity curve that represents a probability distribution.
- The entropic force corresponding to this curve describes an oscillator, whose mass is given by an inverse variance term that measures resistance to change.
- The quantized version of this entropic oscillator has a complex wave function whose squared amplitude gives the probability distribution for price.
- The ground state corresponds to the original propensity curve, which shows the connection between information flows and quantum dynamics.
- The uncertainty of the quantum ground state represents the indeterminate nature of the system, so price changes may reflect not new information, as in the classical model, but rather the *absence* of information.
- While the classical model assumes that market exchanges drive the system to a state of equilibrium, the quantum model suggests they drive it to a state of higher information (lower entropy).
- Excited energy states of the oscillator contribute the skew and kurtosis that characterize financial statistics.
- The model has a number of applications, including as a tool to perform stochastic simulations, or as the basis for a quantum agent-based model.

While as shown above a somewhat similar model can be produced using stochastic differential equations, a distinguishing feature of the quantum version is that price is modeled by a wave function which only collapses to a set value when measured during a transaction. This correctly reflects the indeterminacy of financial systems; incorporates the fact that the measurement procedure affects the system being measured; and forms a natural interface to explore interference effects in cognition, entanglement through social and financial bonds, and the dynamics of excited energy states.

8. Production and consumption

A similar quantum approach can be applied in principle to the production and consumption of goods.⁴³ Suppose that we wish to model a firm which produces some good $x(t)$ at a rate \dot{x} , where the time dependence is suppressed for brevity. Consider first the case of a firm which produces a single item, with a profit function

$$g = p\dot{x} - c(\dot{x}).$$

This represents the money earned per unit time by selling units at a rate \dot{x} and price p , minus the cost of production c . The units of g and c are therefore ET^{-1} (currency over time). The value of \dot{x} which gives maximum profit can be found by setting

$$\frac{\partial g}{\partial \dot{x}} = p - \frac{\partial c(\dot{x})}{\partial \dot{x}} = 0.$$

In neoclassical economics, companies are generally assumed to be operating at this point, which doesn't allow for dynamics, or the fact that production won't usually be at an optimal level. However we can interpret this term $s = \frac{\partial g}{\partial \dot{x}}$ as a correcting force which includes costs and is directed towards optimum profitability. We then write $s = m_s \ddot{x}$, where \ddot{x} is the rate of change of production, and the mass term m_s now represents the inertia of the firm towards that change. If we assume that the inertial mass remains constant (which it need not) then the work performed by the force is

$$\Delta E = \int_0^t s dx = m_s \int_0^t \ddot{x} dx = m_s \int_{x_0}^{x_t} \dot{x} d\dot{x} = \frac{m_s}{2} (\dot{x}_t^2 - \dot{x}_0^2)$$

where \dot{x}_0 and \dot{x}_t represent the initial and final production rates. We can therefore identify the terms of the form $\frac{1}{2} m_s \dot{x}^2$ as the economic equivalent of kinetic energy. The potential energy for a firm is the difference between this, and the maximum obtainable kinetic energy at the optimal level of productivity.

For a consumer, we can similarly write the “profitability” w from the purchase of a stream of goods x as

$$h = -p\dot{x} + u(\dot{x})$$

and the corresponding force towards that purchase as

$$\frac{\partial h}{\partial \dot{x}} = -p + \frac{\partial u(\dot{x})}{\partial \dot{x}}$$

where u represents the utility (see discussion of the demand case in the preceding section).

The complete production/consumption system can then be coupled by assuming that price is

⁴³ Dannenberg, A.A., Estola, M., and Dannenberg, A. (2017). A dynamic theory of economics: What are the market forces?

adjusted dynamically in order to match supply with demand. The dynamics are determined from the force equations for production, consumption, and supply and demand:

$$\begin{aligned} m_s \ddot{x}_s &= p - \frac{\partial c(\dot{x}_s)}{\partial \dot{x}_s} \\ m_d \ddot{x}_d &= -p + \frac{\partial u(\dot{x}_d)}{\partial \dot{x}_d} \\ \dot{p} &= k(\dot{x}_d - \dot{x}_s). \end{aligned}$$

The equations are easily generalised to represent an arbitrary number of firms producing multiple goods. The main difference between the resulting model and the neoclassical version is that the latter sets the mass terms to zero in the equations for production and consumption, and assumes that the third equation representing the imbalance between supply and demand is equal to zero. As Dannenberg et al. (2017) demonstrate, the act of changing from the static neoclassical framework to a dynamical framework is in itself sufficient to reproduce some aspects of phenomena such as economic crises (see their paper for simulations). The equations can also be quantized as before, with the difference that there are now extra force terms which account for costs and consumer desires. The quantum model could simulate entanglement through the introduction of coupling terms.⁴⁴ The inclusion of money and debt would also allow one to model the effects of credit, which is what permits companies (e.g. Tesla) to operate for long periods on borrowed funds. The complexity of the resulting model would probably limit its application to anything but highly simplified situations; however as discussed further below targeted models could be useful for exploring a range of economic phenomena.

As a final note, one can also write the equations in the form

$$\begin{aligned} \dot{v}_s &= \frac{1}{m_s} \left(p - \frac{\partial c(v_s)}{\partial v_s} \right) \\ \dot{v}_d &= \frac{1}{m_d} \left(-p + \frac{\partial u(v_d)}{\partial v_d} \right) \\ \dot{p} &= k(v_d - v_s) \end{aligned}$$

where $v_d = \dot{x}_d$ and $v_s = \dot{x}_s$. These compare directly with the equations used in systems biology to describe the interactions of three chemical species. In biology, stochastic effects

⁴⁴ Kim, YS and Noz, ME (2005). Coupled oscillators, entangled oscillators, and Lorentz-covariant harmonic oscillators. *Journal of Optics B: Quantum and Semiclassical Optics* 7(12): S458.

occur because key reactions, such as the production of mRNA, often involve only a small number of molecules, so reactions are described using the same type of Poisson process.⁴⁵ Results for those models can therefore be carried over to economics. For example biological systems sometimes include feedback loops that limit stochastic fluctuations⁴⁶ and it would be interesting to compare their function with human attempts to smooth economic fluctuations.

9. Entanglement

As discussed in the book, a key advantage of the quantum approach in economics – but one which to my knowledge has not previously been addressed by researchers in quantum finance – is that it provides a natural framework for thinking about financial entanglement through loans and derivatives.

To first motivate the discussion, consider the physical example of a pair of entangled electrons, denoted 1 and 2, each of which has spin $\frac{1}{2}$ when measured along a particular axis, but in opposite directions. The spin part of their wave function can be written as a superposition of two states:

$$|S\rangle = \frac{1}{\sqrt{2}} |1 \uparrow\rangle|2 \downarrow\rangle - \frac{1}{\sqrt{2}} |1 \downarrow\rangle|2 \uparrow\rangle$$

where the arrow indicates the direction of spin of each electron.

The wave function tells us nothing about the direction of spin for either electron, only that they are opposite, so the total spin is zero. Now, suppose that we measure the spin for electron 1. We would expect an equal chance of getting a positive or negative result. If it is the former, then the system must have collapsed to an eigenstate with positive eigenvalue, so is of the form

$$|S\rangle = |1 \uparrow\rangle|2 \downarrow\rangle$$

A measurement of particle 2 can now yield only a negative result. The reason is that the wave function describes the system, including both particles, so a measurement on one is equivalent to a measurement on the system as a whole.

⁴⁵ Ramsey, S., Orrell, D., & Bolouri, H. (2005), ‘Dizzy: stochastic simulation of large-scale genetic regulatory networks’, *Journal of bioinformatics and computational biology* 3 (02), 415-436.

⁴⁶ Ramsey, S., et al. (2006), ‘Dual feedback loops in the GAL regulon suppress cellular heterogeneity in yeast’, *Nature genetics* 38(9): 1082.

The financial version of entanglement can be expressed using a similar formalism. Instead of two entangled electrons, consider two people entangled by a loan contract; and instead of spin direction, we will use loan status (i.e. “default” or “no default”). As in quantum cognition, the debtor is modelled as initially being in a superposition of two states, with a decision acting as a measurement event. The loan status can therefore be expressed by a wave function of the form:

$$|S\rangle = \alpha |1 \uparrow\rangle|2 \downarrow\rangle - \beta |1 \downarrow\rangle|2 \uparrow\rangle$$

Here α^2 and β^2 add to 1, and give the probability of default $|1 \uparrow\rangle|2 \downarrow\rangle$ and no default $|1 \downarrow\rangle|2 \uparrow\rangle$ respectively, so reflect the debtor’s propensity to default at a particular moment. If the debtor decides to default on the loan, that acts as a measurement on the system as a whole. At any time after that, if the creditor decides to assess the state of the loan, the result can only indicate default. The two parties are thus entangled.

Of course, systems can be correlated without any need to invoke quantum effects.⁴⁷ However the key point is that we are treating the debtor’s state regarding the loan as being in a superposition of the two states “default” and “no default”. The state of the loan is therefore indeterminate (we don’t know whether the debtor will default) yet still correlated, which is the essence of entanglement.

Another possible objection is that, after one of a pair of entangled particles has been measured, the second doesn’t need to check with the first to find out what its state is; while with a loan the creditor does. However the wave function equation applies to the loan agreement, which is an abstract thing that encompasses both parties. So from the point of view of that wave function (which again is what we are modelling) the state does change instantaneously; it is only measurements that take time. The difference between the physics version, and the financial version, then reduces to a question of the nature and reality of such wave functions, which would depend on one’s interpretation of quantum theory, and is a topic of debate for both physicists and social scientists.⁴⁸ But from a mathematical modelling perspective the two are the same.

⁴⁷ For example, suppose I have two beads, one red and one blue, and I give one to a friend without looking. Then if I check and find that I have the red one, I know that my friend has the blue one.

⁴⁸ A widely discussed example is whether Xantippe, the wife of Socrates, became a widow the instant her husband was forced to commit suicide, or only when she found out later. See: Wendt, A. (2015), *Quantum Mind and Social Science: Unifying Physical and Social Ontology* (Cambridge: Cambridge University Press), p. 194.

One feature of the system is that, unlike for electrons, there is now only one axis of measurement. This means that the behaviour of a loan agreement is much less subtle than the physical version (though some social scientists do argue for rich versions of mental entanglement based on physical principles); and also that it is not possible to reproduce Bell-type experiments, where entanglement is tested by changing the orientation of the axis. However Bell's experiments do not define entanglement, but were devised as a way to tease out entanglement for systems that cannot be queried more directly. For loans, the entanglement is encoded by the terms of the agreement. Again, the equation applies only to the loan agreement, so default may for example be followed by a complex negotiation, but the same is true in a physical system where other forces can also come into play.

Since most money is produced through private bank lending, and the financial system is dominated by complex derivatives contracts, financial entanglement is a tremendously important part of the economy, yet one which has been largely neglected in mainstream models, precisely because they are based on a classical atomistic paradigm. A number of techniques are currently being developed to simulate collective decision-making using a quantum approach, and these could be used to model phenomena such as mass defaults, or the impact of collective behaviour on the generation of credit in an economy.⁴⁹

To illustrate this point, we can return to the case of strategic default. The quantum analysis of preference reversal was based on the idea that subjective attitudes are entangled within the mind of the decision-maker, and so resolve themselves in a manner that is context-dependent. However entanglement also operates at the social level, for example as a result of the exchange of information between the society members.

To accommodate social effects, Yukalov and Sornette⁵⁰ extend the decision space to be the tensor product

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_S.$$

Here $\mathcal{H}_A = \text{Span}\{A_i B_j\}$ is the decision space for the individual, and \mathcal{H}_S represents the decision space for the rest of society, which can similarly be expressed as the tensor product

⁴⁹ One researcher investigating quantum models of collective decision making is Michael Schnabel: <https://harris.uchicago.edu/directory/michael-schnabel>

⁵⁰ Yukalov VI & Sornette D (2015), Role of Information in Decision Making of Social Agents, *International Journal of Information Technology & Decision Making* 14(05): 1129-1166.

of the individual spaces of the other society members. The state of the society is represented by the statistical operator $\hat{\rho}_{AB}$. This is normalised so that

$$\text{Tr}_{AB} \hat{\rho}_{AB}(\mu) = 1$$

where the trace operation is performed over \mathcal{H} , and μ represents an amount of information.

By examining how the prospect probabilities evolve as a function of the information level μ , Yukalov and Sornette then show formally that the net effect of interaction between the individual and society is to attenuate the individual's attraction function, so that

$$\log_{\mu \rightarrow \infty} p(\pi_j, \mu) = f(\pi_j)$$

In other words, as information increases, the probability converges to the classical utility factor. A similar approach was used by (Khrennikova & Haven, 2016) to model how voter preferences are shaped by the informational “bath” generated by mass media.

Indeed, experimental evidence shows that cognitive biases tend to reduce when participants exchange information by consulting with others (Charness et al., 2010). For the case considered above of mortgage default, we would expect the strategic default rate to increase with information flow. This was again confirmed by the data, which showed that people who know someone who has strategically defaulted are 82% more likely to declare their intention to default: as (Guiso et al., 2013: 1514) write, “This effect does not seem to be due to clustering of people with similar attitudes, but rather to learning about the actual cost of default. We find a similar learning effect from exposure to the media, an effect that is reduced when the media start to cover the topic more extensively.”

According to an estimate from First American, it would have cost some \$745 billion, or slightly more than the size of the 2008 bank bailout, to restore the lost equity of all underwater borrowers (Streitfeld, 2010). These entanglements were further extended and amplified through the use of complex derivatives such as collateralized mortgage obligations, which were held internationally. Prior to the crisis, these were seen as having a stabilising effect on the economy. As the International Monetary Fund (2006: 51) noted: “The dispersion of credit risk by banks to a broader and more diverse group of investors, rather than warehousing such risks on their balance sheets, has helped to make the banking and overall financial system more resilient.” Bernanke (2006) echoed the IMF when he said that “because of the dispersion of financial risks to those more willing and able to bear them, the

economy and financial system are more resilient.” These conclusions were apparently not based on modelling, since the models used at the time didn’t even include a banking sector. Financial entanglement is therefore a dominant factor in the economy, but also one of the least understood.

10. Summary

The quantum approach has successfully been used to model the economy at both the level of individuals (in quantum cognition) and the level of markets (quantum finance). In either case, the state of the system is represented using a Hilbert space. Measurement procedures such as decisions and transactions take precedence over internal states such as known preferences or inherent values. The quantum approach therefore differs fundamentally from the classical one, and can be extended to offer an alternative model of the economy in general.

While the aim of this document is only to give an idea of how quantum techniques can be applied to the economy, the literature in this area is quite large and different researchers take different approaches. As shown by empirical results in quantum cognition, the quantum approach appears to be a natural fit for modelling human decision-making. And while quantum finance has not been widely adopted by the quantitative finance community, some traders have adopted the quantum methodology to understand and predict for example the behaviour of illiquid assets.

From the larger perspective of quantum economics, a main advantage of the quantum approach is that it naturally incorporates the dualistic properties of money. In classical economics, price is essentially equated with value (with allowances for “market failures”). In quantum mechanics, prices are seen as emerging from monetary transactions. One consequence is to sever the direct link between price and value. Another is to concentrate the modeller’s attention on the entangling properties of money.

A natural extension of the market models considered above, and an interesting longer-term research project, would be a quantum agent-based model of something like a housing market. Following the approach used to model the propensity to buy or sell individual stocks (Section 7), each house could be considered as a separate single-asset market. Buyers and sellers would be entangled to a degree with each other, and to the news flow which could be

modelled as a quantum variable, and to the financial markets through loans. Such a model could simulate the kind of market contagion seen in housing markets, such as “fear of missing out” when prices are rising. It could also include the process of money creation through private lending, which grows the money supply and leads to asset price inflation.

To summarise, the quantum approach offers a natural framework for modelling key economic properties such as indeterminacy and entanglement. It also explicitly accounts for stochastic dynamic effects, of the sort that are regularly studied in areas such as systems biology, but have played a much smaller role in economics (apart from finance). While many people with a background in physics will be familiar with the quantum approach, and can easily apply methods from e.g. statistical mechanics to derive results, those trained in a classical approach may at first find it awkward or overly elaborate. However one of the main lessons of quantum economics is that, just because the economy *emerges* from quantum effects, this does not imply that quantum models are always obligatory. The complex behaviour of water, which ultimately arises from quantum properties, may drive the weather system but is not part of weather models; and similarly it is possible to simulate the flow of money in a way that respects its complex emergent properties without needing to go down to the quantum level. The quantum approach can also be used to rule out certain modelling approaches, including Dynamic Stochastic General Equilibrium models (the so-called workhorses of macroeconomics), which rely on classical assumptions such as equilibrium.

For more background and further reading, see davidorrell.com/quantumresources.html

Contact the author at: futureofeverything.wordpress.com/about/